DOUBLE SEQUENCES AND SERIES

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "DOUBLE SEQUENCES AND SERIES", submitted by VINCENT W. S. LEUNG, B.SC., in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

The study of convergence and summability of simple sequences and series has aroused the interest of mathematicians since the beginning of the eighteenth century. However, it was not until the dawn of the twentieth century that mathematicians began their investigation of double and multiple sequences and series and extend the results of simple sequences and series to sequences and series with multiplicity greater than one.

The aim of this thesis is to make an extensive survey of all the literature written on double sequences and series under the topics convergence, summability by general methods, summability by special methods, and other results related to double sequences and series.

A comparative study of the five definitions of convergence is made, and, wherever possible, examples are cited to illustrate the usefulness of each definition and their mutual relations. Necessary and sufficient conditions are given for which the matrix (triangular or square) transformation may transform an arbitrary sequence into one of the given types of sequences. This is followed by a brief description of the various special methods of summation for double series. Finally, a bibliography of the articles on other topics in double sequences and series is listed.

Except for some special cases, proofs of the theorems are omitted and references are given to which the reader may refer.

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CHAPTER I

CONVERGENCE OF DOUBLE SEQUENCES AND SERIES

0. Introduction

A double sequence is a function s on $J^+ \times J^+$ [or on $(J^+ \cup \{0\}) \times (J^+ \cup \{0\})]$, whose value at (m, n) is most commonly denoted by s_{mn} ... in which case the double sequence is also written as $\{s_{mn}\}_{m,n=1}^{\infty}$ (or $\{s_{mn}\}_{m,n=0}^{\infty}$) .

If s is a double sequence to a topological space X, then s is convergent to σ if $\sigma \in X$ and there corresponds to each open set $U \subset X$ for which $\sigma \in U$, a $k \in J^+$ such that $s_{mn} \in U$ whenever $m, n \in J^+$ and $m, n > k$ ... in which case we write

$$\lim_{m,n \to \infty} s_{mn} = \sigma$$

or

$$s_{mn} \to \sigma \quad \text{as} \quad m,n \to \infty ,$$

Such a double sequence is convergent if it is convergent to some σ and otherwise it is divergent.

A double series is an indicated sum $\sum_{m,n=1}^{\infty} a_{mn}$ (or $\sum_{m,n=0}^{\infty} a_{mn}$) of a double sequence to a set X which is closed under addition.

Some of the properties with which we shall be concerned in this thesis are valid in the case where X is a topological group under

addition or a topological vector space. However, we shall restrain ourselves and consider only the case where X is (the metric vector space) Co [(1)].

1. Types of Convergence

In the following we give the various types of convergence of double sequences and series that are in existence.

A. $\mathcal{F}$-convergence [(*)]

Let $\mathcal{F}$ be a collection of finite subsets of $J^+ \times J^+$ for which there is a sequence $\{F_n\}_{n=1}^{\infty} \subset \mathcal{F}$ such that $F_n \subset F_{n+1}$ $(n \in J^+)$ and

$$\bigcup_{n=1}^{\infty} F_n = J^+ \times J^+ .$$

The double series $\sum_{m,n=1}^{\infty} a_{mn}$ is $\mathcal{F}$-convergent to α if $\alpha \in \text{Co}$ and there corresponds to each $\epsilon > 0$ a $k \in J^+$ such that

$$\Big| \sum_{(m,n)\in F} a_{mn} - \alpha \Big| < \epsilon$$

whenever $F \in \mathcal{F}$ and $F \supset F_k$; it is $\mathcal{F}$-convergent if it is $\mathcal{F}$-convergent to some $\alpha \in \text{Co}$ and otherwise it is $\mathcal{F}$-divergent. The most frequently encountered collections $\mathcal{F}$ are the sets of all

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(1) Extract of the lecture notes given by Professor A. E. Livingston at the University of Alberta, Edmonton, Canada, 1965.

(*) Result of Livingston as in (1).

rectangles $R_{mn} = J_m \times J_n \qquad (m, n \in J^+)$,

triangles $T_n = \{(k, \ell) \in J^+ \times J^+ \mid k + \ell \leq n\}$, $(n \in J^+)$,

hyperbolas $H_n = \{(k, \ell) \in J^+ \times J^+ \mid k\ell \leq n\}$, $(n \in J^+)$.

The corresponding sets F_n being (the squares) $J_n \times J_n$, T_n , and H_n respectively; we speak then of Pringsheim, Cauchy and Dirichlet convergence and divergence, respectively.

B. Global-convergence [27, 1940].

A double sequence $\{s_{mn}\}_{m,n=1}^{\infty}$ is said to be globally convergent if to each $\epsilon > 0$ corresponds an integer N such that

$$|s_{mn} - s_{m+p,n} - s_{m,n+q} + s_{m+p,n+q}| < \epsilon$$

whenever $m, n > N$ and $p, q \geq 0$.

C. σ- sum convergence (Amerio convergence).

We describe the σ-sum convergence via the following definitions.

Definition 1.1.1 [2, 1941] and [33, 1945].

A partial sum is said to be a sigma-sum (or equivalently, σ-sum) if it has the following property: if it contains a_{pq} then it also contains the rectangular sum

$$R_{pq} = \sum a_{ij} \qquad (0 \leq i \leq p \ , \ 0 \leq j \leq q) \ .$$

Definition 1.1.2

Given indices (p, q), a σ-sum is said to be a σ-sum relative to (p, q) [written σ-sum (p, q)] if it contains a_{pq}.

Definition 1.1.3

The series $\sum_{i,j=1}^{\infty} a_{ij}$ converges to the sum α if to every $\epsilon > 0$ there correspond indices (p, q) such that

$$|\sigma - \alpha| < \epsilon$$

for every σ that is a σ-sum (p, q).

Remark 1.1.1

The above definition of convergence was proposed independently by Amerio [2, 1941] and Sheffer [33, 1945]. The σ-sum convergence is essentially $\mathcal{F}$-convergence with the additional condition that the collection $\mathcal{F}$ of all finite subsets F of $J^+ \times J^+$ for which

$$(p, q) \in F \Rightarrow \{(i, j) | i \leq p \text{ , } j \leq q\} \subset F \text{ .}$$

For the sake of completeness, we treat it as an independent method calling it σ-convergence in the later sections. This new method provides a theory more nearly analogous to that of simple series than is obtained by Pringsheim-convergence. (For a comparison of Pringsheim - σ - and Regular-convergence, see later sections).

D. Regular-convergence [36, 1947]

Definition 1.1.4

The series $\sum_{i,j=1}^{\infty} a_{ij}$ is regularly convergent if it is Pringsheim-convergent and if every row and column converges.

E. A-convergence (2)

Definition 1.1.5

The series $\sum_{i,j=1}^{\infty} a_{ij}$ is said to be A-convergent if at least one of the simple series into which the original series can be arranged, is convergent.

2. Characterizations of Convergence

A. $\mathcal{F}$-convergence (*)

Theorem 1.2.1

If $\sum_{m,n=1}^{\infty} a_{mn}$ is $\mathcal{F}$-convergent to α, then

$$\lim_{k \to \infty} \sum_{(m,n)\in F_k} a_{mn} = \alpha .$$

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(2) This definition is due to Hitotumatu, Sin., On the convergence of a multiple power series. Kōdai Math. Sem. Rep. (1952), 111-114.

(*) See Livingston (1).

Theorem 1.2.2

If $\sum_{m,n=1}^{\infty} a_{mn}$ is $\mathcal{F}$-convergent, then

$$\lim_{n \to \infty} \sum_{(k,\ell)\in F_n - F_{n-1}} a_{k\ell} = 0 .$$

Theorem 1.2.3

If $\sum_{m,n=1}^{\infty} a_{mn}$ is Pringsheim-convergent, then $\lim_{m,n \to \infty} a_{mn} = 0$.

Remark 1.2.1

A double series may be $\mathcal{F}$-convergent without having $\lim_{m,n \to \infty} a_{mn} = 0$ as is shown in the following example.

Example 1.2.1

Let $a_{i,i-1} = -a_{i-1,i} = 1$ for $i = 2, 3, 4, \ldots$ and $a_{ij} = 0$ otherwise for $i, j \in J^+$. Then the double series $\sum_{i,j=2}^{\infty} a_{ij}$ is $\mathcal{F}$-convergent (i.e. both Cauchy-and Dirichlet-convergent) and $\lim_{i,j \to \infty} a_{ij}$ is not equal to zero.

<u>Theorem 1.2.4</u>

Let $\sum_{m,n=1}^{\infty} a_{mn}$ be $\mathcal{F}$-convergent to α .

(i) If $c \in Co$, then $\sum_{m,n=1}^{\infty} ca_{mn}$ is $\mathcal{F}$-convergent to $c\alpha$.

(ii) If $\sum_{m,n=1}^{\infty} b_{mn}$ is $\mathcal{F}$-convergent to β , then

$\sum_{m,n=1}^{\infty} (a_{mn} + b_{mn})$ is $\mathcal{F}$-convergent to $\alpha + \beta$.

<u>Theorem 1.2.5</u>

If $a_{mn} \geq 0$ for $m, n \in J^+$, then $\sum_{m,n=1}^{\infty} a_{mn}$ is $\mathcal{F}$-convergent

if and only if

$$\left\{ \sum_{(m,n)\in F} a_{mn} \;\middle|\; F \in \mathcal{F} \right\}$$

is bounded, in which case

$$\sum_{m,n=1}^{\infty} a_{mn} = \text{lub} \left\{ \sum_{(m,n)\in F} a_{mn} \;\middle|\; F \in \mathcal{F} \right\} .$$

<u>Corollary</u>

If $a_{mn} \geq 0$ for $m, n \in J^+$, then $\sum_{m,n=1}^{\infty} a_{mn}$ is $\mathcal{F}$-convergent

to α if and only if

$$\lim_{k \to \infty} \sum_{(m,n) \epsilon F_k} a_{mn} = \alpha .$$

Theorem 1.2.6

If $a_{mn} \geq 0$ for $m, n \epsilon J^+$, then the double series $\sum_{m,n=1}^{\infty} a_{mn}$ is $\mathcal{F}$-convergent to α if and only if $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn}$ is convergent to α .

Definition 1.2.1

$\sum_{m,n=1}^{\infty} a_{mn}$ is $\mathcal{F}$-absolutely convergent if $\sum_{m,n=1}^{\infty} | a_{mn} |$ is $\mathcal{F}$-convergent.

Theorem 1.2.7

An $\mathcal{F}$-absolutely convergent double series is $\mathcal{F}$-convergent.

Remark 1.2.2

Theorems 1.2.5 and 1.2.6 show that we need not specify $\mathcal{F}$ when $\sum_{m,n=1}^{\infty} a_{mn}$ is $\mathcal{F}$-absolutely convergent. Furthermore, in any argument about $\mathcal{F}$-absolute convergence, we may use whichever collection $\mathcal{F}$ is most convenient for the series at hand.

Agreement

We will say that $\sum_{m,n=1}^{\infty} a_{mn}$ is absolutely convergent whenever it is $\mathcal{F}$-absolutely convergent for some $\mathcal{F}$.

Theorem 1.2.8

If $\sum_{m,n=1}^{\infty} a_{mn}$ is absolutely convergent, then $\lim_{m,n \to \infty} a_{mn} = 0$.

Example 1.2.2

$\sum_{m,n=1}^{\infty} r^{m+n}$, r fixed, is absolutely convergent if $|r| < 1$, but is not absolutely convergent if $|r| \geq 1$.

Example 1.2.3

$\sum_{m,n=1}^{\infty} r^{mn}$, r fixed, is absolutely convergent if $|r| < 1$, but is not absolutely convergent if $|r| \geq 1$.

B. Global-convergence [27, 1940]

Theorem 1.2.9

A bounded sequence of which each row and each column is convergent, is Pringsheim-convergent if and only if it is globally convergent.

Theorem 1.2.10

Each bounded globally convergent sequence contains a convergent double subsequence.

Remark 1.2.3

The particular sequence $s_{mn} = (-1)^m + (-1)^n$, which is bounded and globally convergent, contains convergent double subsequences converging to different values.

C. σ-convergence [2, 1941], [33, 1945]

Theorem 1.2.11

A necessary and sufficient condition that the series

$\sum_{m,n=1}^{\infty} a_{mn}$ σ-converges is that to every $\epsilon > 0$ there

correspond indices (p, q) such that

(1.2.1) $$|\sigma' - \sigma''| < \epsilon$$

for every pair σ', σ'' of σ-sums (p, q).

Proof

First suppose that the series converges, to sum α. Then given $\epsilon > 0$, indices (p, q) exist so that $|\alpha - \sigma'| < \epsilon/2$, $|\alpha - \sigma''| < \epsilon/2$ whenever σ', σ'' are σ-sums (p, q). Hence (1.2.1) holds. Now suppose (1.2.1) holds; to show that the series σ-converges. If σ' is fixed, (1.2.1) shows

that the set $\{\sigma''\}$ of all σ-sums (p, q) is bounded, and therefore has at least one limit point α. Accordingly, there is a σ-sum (p, q) say σ''', such that $|\alpha - \sigma'''| < \epsilon$; and from (1.2.1), $|\alpha - \sigma| < 2\epsilon$ for every σ-sum (p, q). From the arbitrariness of ϵ we conclude that $\sum_{m,n=1}^{\infty} a_{mn}$ converges to α.

<u>Theorem 1.2.12</u>

If $\sum_{m,n=1}^{\infty} a_{mn}$ σ-converges, then to every $\epsilon > 0$ correspond indices (p, q) such that

$$|a_{mn}| < \epsilon$$

for every a_{mn} not in R_{pq}.

<u>Corollary</u>

If $\sum_{m,n=1}^{\infty} a_{mn}$ σ-converges, then $\lim_{m,n \to \infty} a_{mn} = 0$.

<u>Remark 1.2.4</u>

Theorem 1.2.12 is similar to Theorem 1.2.3. An example showing a double series which is Pringsheim-convergent but not σ-convergent will be given in a later section.

Theorem 1.2.13

Let $\sum_{m,n=1}^{\infty} a_{mn}$ σ-converge to the sum α. If m (if n) is fixed, the resulting row series (column series) converges; and if its sum is denoted by $\alpha_{m.}$ (by $\alpha_{.n}$), then also the series $\sum_{m=1}^{\infty} \alpha_{m.}$ (series $\sum_{n=1}^{\infty} \alpha_{.n}$) converges, and its sum is α.

Theorem 1.2.14

If the double series $\sum_{m,n=1}^{\infty} a_{mn}$ is σ-convergent, then there exists an $\ell \in J^+$ such that its partial sums

$$s_{\mu\nu} = \sum_{m=1}^{\mu} \sum_{n=1}^{\nu} a_{mn}$$

are uniformly bounded for all $\mu, \nu > \ell$.

Theorem 1.2.15

Given series $\sum_{m,n=1}^{\infty} a_{mn}$. If for each fixed m the simple series

$$\alpha_{m.} = \sum_{n=1}^{\infty} |a_{mn}|$$

converges, and if $\sum_{m=1}^{\infty} \alpha_{m.}$ converges, then the original

series $\sum_{m,n=1}^{\infty} a_{mn}$ converges absolutely; and correspondingly for columns.

Remark 1.2.5

Theorem 1.2.4 and Theorem 1.2.8 are of course also valid for σ-convergence.

Definition 1.2.2

If the first p rows and q columns are deleted from the series $\sum_{m,n=1}^{\infty} a_{mn}$, the new series, which has a_{pq} as its leading term, will be called the truncate of order (p, q) relative to the original series.

Theorem 1.2.16

If the series $\sum_{m,n=1}^{\infty} a_{mn}$ σ-converges, so does every truncate series.

Remark 1.2.6

Theorem 1.2.16 is false for Pringsheim-convergence, as the following example shows.

Example 1.2.4

Let the first p rows of a double series be

$$1 + 2 + 3 + 4 + 5 + \ldots$$

and the following $(p + 1)^{th}$ to $(2p)^{th}$ rows be

$$(-1) + (-2) + (-3) + (-4) + \ldots$$

and zero otherwise. Then the double series is Pringsheim-convergent to zero but the truncate with the first p rows deleted is divergent.

D. Regular-convergence [36, 1947]

Definition 1.2.3

A function a_{mn} is of bounded variation in (m, n) if

(i) a_{mn} is, for every fixed value of m or n, of bounded variation in n or m.

(ii) The series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{mn} - a_{m,n+1} - a_{m+1,n} + a_{m+1,n+1}|$$

is convergent.

Theorem 1.2.17

If a_{mn} is of bounded variation, and $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn}$ is regularly convergent, then the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \; u_{mn}$$

is regularly convergent.

Theorem 1.2.18

If a_{mn} is of bounded variation and tends regularly to zero,

and $\sum_{m=1}^{\mu} \sum_{n=1}^{\nu} u_{mn}$ is bounded, then the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} u_{mn}$$

is regularly convergent.

Lemma 1.2.1

If $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn}$ is a divergent series of positive terms,

we can find ϵ_{mn} so that

(i) ϵ_{mn} decreases when m or n increases,

(ii) ϵ_{mn} tends regularly to zero, and

(iii) the series $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \epsilon_{mn} c_{mn}$ is divergent.

Proof

(1) Suppose first that at least one row or column of the original series, say the ν^{th} row $\sum_{m=1}^{\infty} c_{m\nu}$, is divergent. By a lemma due to Abel[3], we can choose

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(3) 'Sur les Séries', Oeuvres, vol. 2, pp. 197-205.

a steadily decreasing sequence η_m, with limit zero, so that $\sum \eta_m c_{m\nu}$ is divergent. We take

$$\epsilon_{mn} = \eta_m \quad (n \leq \nu)$$

$$\epsilon_{mn} = 0 \quad (n > \nu),$$

then it is clear that the conditions of the lemma are satisfied.

(2) Suppose that every row and column is convergent; and let

$$\sum_{m=1}^{\infty} c_{mn} = \gamma_n , \quad \sum_{n=1}^{\infty} c_{mn} = \gamma_m .$$

Then $\sum_{m=1}^{\infty} \gamma_m$ is divergent. We choose a steadily decreasing sequence η_m so that $\sum_{m=1}^{\infty} \eta_m \gamma_m$. is divergent. Then $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c'_{mn}$, where

$$c'_{mn} = \eta_m c_{mn} ,$$

is divergent; and so $\sum_{n=1}^{\infty} \gamma'_n$, where

$$\gamma'_n = \sum_{m=1}^{\infty} \eta_m c_{mn} ,$$

is divergent. We now choose a steadily decreasing sequence ξ_n, with limit zero, so that

$\sum_{n=1}^{\infty} \xi_n \gamma'_n$ is divergent. It is clear that if we write

$$c''_{mn} = \eta_m \xi_n c_{mn} = \epsilon_{mn} c_{mn} ,$$

all the conditions of the lemma will be satisfied.

Lemma 1.2.2

If $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn}$ is a divergent series of positive terms, we can choose a sequence of pairs of integers (m_i, n_i) , tending to infinity with i , so that the series

$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c'_{mn}$ is divergent where $c'_{mn} = 0$ if $m = m_i$, $n \leq n_i$ or $m \leq m_i$, $n = n_i$ and $c'_{mn} = c_{mn}$ otherwise.

Theorem 1.2.19

If $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} u_{mn}$ is regularly convergent whenever $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn}$ is regularly convergent, then a_{mn} is of bounded variation.

E. A-convergence

Due to the restrictiveness of its definition, the A-convergence is seldom applied to double series in general.

3. Comparison and Generalization of the Methods of Convergence

Remark 1.3.1

Pringsheim-convergence is not equivalent to Cauchy or Dirichlet-convergence. In the case of Cauchy and Dirichlet-convergence we may conclude that the series $\sum_{m,n=1}^{\infty} a_{mn}$ is $\mathcal{F}$-convergent to α if and only if

$$\lim_{k\to\infty} \sum_{(m,n)\in F_k} a_{mn} = \alpha \; .$$

But $\lim_{k\to\infty} \sum_{(m,n)\in F_k} a_{mn} = \alpha$ does not necessarily imply that $\sum_{m,n=1}^{\infty} a_{mn}$ is Pringsheim-convergent.

Example 1.3.1

The series

$$\sum_{m,n=1}^{\infty} (-1)^{m+n} \Big/ [\ln(m+1)\,\ln(n+1)]$$

is Pringsheim-convergent, but neither Cauchy- nor Dirichlet-convergent.

Example 1.3.2

If $a_{i,i-1} = -a_{i-1,i} = 1$ for $i = 2, 3, 4, \ldots$ and $a_{ij} = 0$ otherwise for $i, j \in J^+$, then $\sum_{m,n=1}^{\infty} a_{mn}$ is both Cauchy-and Dirichlet-convergent but not Pringsheim convergent.

Remark 1.3.2

Each double sequence which is Pringsheim-convergent is globally convergent, but the converse is not true.

Remark 1.3.3

If a double series converges in the sense of Pringsheim, the simple series forming the terms of a given row or column need not converge.

Example 1.3.3

The double series with the following terms

$$0! + 1! + 2! + \ldots + n! + \ldots$$

$$-0! + -1! - 2! - \ldots - n! - \ldots$$

$$0 + 0 + 0 + \ldots$$

is Pringsheim-convergent to 0 but each of the two rows is divergent.

Remark 1.3.4

A σ-convergent series is also Pringsheim-convergent (and to the same sum), but not conversely.

Example 1.3.4

The series in Example 1.3.3 is Pringsheim-convergent but not σ-convergent.

Remark 1.3.5

The Pringsheim-convergent series whose first two rows are

$$\begin{array}{cccccccc} +1 & -1 & +1 & -1 & +1 & -1 & + & \cdots \\ -1 & +1 & -1 & +1 & -1 & +1 & - & \cdots \end{array}$$

and whose lower rows are series of zero is not σ-convergent. Thus σ-convergence is weaker than Pringsheim-convergence. On the other hand, σ-convergence is stronger than absolute convergence.

Remark 1.3.6

If $\sum\limits_{m,n=1}^{\infty} a_{mn}$ converges absolutely either in the Pringsheim sense or in the sense of σ-convergence, it converges absolutely by the other.

Remark 1.3.7

For double series of positive terms, $\mathcal{F}$-convergence, (and therefore σ-convergence).

Remark 1.3.8

An absolutely convergent double series is A- , $\mathcal{F}$- and σ-convergent but the converse is false as is illustrated below.

<u>Example 1.3.5</u>

The series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

$$+ (-1) + (-\frac{1}{2}) + (-\frac{1}{3}) + (-\frac{1}{4}) + \ldots$$

$$+ 0 + 0 + 0 + 0 + \ldots$$

is A- and $\mathfrak{I}$ -convergent but not absolutely convergent.

<u>Remark 1.3.9</u>

In general, A- and $\mathfrak{I}$ - or σ-convergence are not the same.

<u>Example 1.3.6</u>

The series

$$0 + 1 + \frac{1}{3} + \frac{1}{5} + \ldots$$

$$+ (-\frac{1}{2}) + 0 + 0 + 0 + \ldots$$

$$+ (-\frac{1}{4}) + 0 + 0 + 0 + \ldots$$

$$+ (-\frac{1}{6}) + 0 + 0 + 0 + \ldots$$

$$+ \ldots$$

is A-convergent but not σ-convergent.

<u>Example 1.3.7</u>

The series

$$-1 + 1 + 2 + 3 + 4 + \ldots$$

$$+1 + (-1) + (-2) + (-3) + (-4) + \dots$$

$$+2 + (-2) + 0 + 0 + 0 + \dots$$

$$+3 + (-3) + 0 + 0 + 0 + \dots$$

is Pringsheim-convergent but not A-Convergent.

Remark 1.3.10

The terms of A-convergent series are bounded; but this is not true for Pringsheim-convergence series. To illustrate the importance of such a difference, we consider the following result.

Lemma 1.3.1

If the terms of a double power series

$$\sum_{m,n=0}^{\infty} a_{mn} x^m y^n$$

are bounded at $x = x_o$, $y = y_o$, or especially if the power series is A-convergent at $x = x_o$, $y = y_o$, then it converges uniformly and absolutely in every compact subset contained in $|x| < |x_o|$, $|y| < |y_o|$.

Note

The assumption of the lemma cannot be replaced by the Pringsheim-convergence at $x = x_o$, $y = y_o$, for Pringsheim-convergence does not imply the boundedness of the terms of the series.

Remark 1.3.11

Regular-convergence is more powerful than σ-convergence, i.e. every series which is σ-convergent is also regularly convergent (to the same sum) but not conversely.

Remark 1.3.12

Regular-convergence is equivalent to the following modification of σ-convergence. The σ-convergent series $\sum_{m,n=0}^{\infty} a_{mn}$ is Pringsheim-convergent to α with α satisfying the equality

$$\alpha = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} .$$

By combining the definitions of convergence we obtain the following results.

Theorem 1.3.1 (*)

If the series $\sum_{m,n=1}^{\infty} a_{mn}$ is Pringsheim-convergent to α and $\sum_{n=1}^{\infty} a_{mn}$ is convergent to $a_{m.}$ for each $m \in J^+$, then

(*) See Livingston (1).

$\sum_{m=1}^{\infty} a_{m.}$ is convergent to α (i.e.

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} = \sum_{m,n=1}^{\infty} a_{mn} \Big) .$$

Remark 1.3.13

Note that Pringsheim-convergence cannot be replaced by Cauchy- or Dirichlet-convergence.

Example 1.3.8

If $a_{i,i-1} = -a_{i-1,i} = 1$ for $i = 2, 3, 4, \ldots$ and $a_{ij} = 0$ otherwise for $i, j \in J^+$, then $\sum_{m,n=1}^{\infty} a_{mn}$ is Cauchy-(and Dirichlet-) convergent to 0. But $\sum_{n=1}^{\infty} a_{mn}$ is convergent to -1 for $m = 1$ and to 0 for $m > 1$, so that $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} = -1$. Thus the above theorem is false for Cauchy-(or Dirichlet-) convergence.

Example 1.3.9

If $a_{mn} = [\sin n - \sin(n-1)] / [(m-1)m]$, with $\frac{1}{(m-1)m} = -1$ for $m = 1$. Then

$$\sum_{n=1}^{N} a_{mn} = \frac{\sin N}{m(m-1)} \quad \text{and} \quad \sum_{m=1}^{M} \sum_{n=1}^{N} a_{mn} = -\frac{\sin N}{M}$$

so that $\sum_{m,n=1}^{\infty} a_{mn}$ is Pringsheim-convergent to zero. But $\sum_{n=1}^{\infty} a_{mn}$ is divergent for each $m \in J^+$. Thus, the hypothesis that $\sum_{n=1}^{\infty} a_{mn}$ is convergent for each $m \in J^+$ cannot be relaxed in Theorem 1.3.1.

Notation

We write

$$\sum_{m=k,n=\ell}^{\infty} a_{mn}$$

for the double series

$$\sum_{m,n=1}^{\infty} a_{k-1+m,\ell-1+n} .$$

Theorem 1.3.2 (*)

Suppose that each $F \in \mathcal{F}$ has the property: If $(m, n) \in F$, then (m, k), $(\ell, n) \in F$ for $k \in J_n$ and $\ell \in J_m$. If $M, N \in J^+$, $\sum_{n=1}^{\infty} a_{mn}$ is convergent for each $m \in J_{M-1}$, and $\sum_{m=1}^{\infty} a_{mn}$ is convergent for each $n \in J_{N-1}$, then $\sum_{m=M,n=N}^{\infty} a_{mn}$ is $\mathcal{F}$-convergent if and only if $\sum_{m,n=1}^{\infty} a_{mn}$

(*) See Livingston (1).

is $\mathcal{J}$-convergent, in which case

$$\sum_{m,n=1}^{\infty} a_{mn} = \sum_{m=M,n=N}^{\infty} a_{mn} + \sum_{m=1}^{M-1} \sum_{n=1}^{\infty} a_{mn} + \sum_{n=1}^{N-1} \sum_{m=1}^{\infty} a_{mn} - \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} a_{mn} .$$

Corollary

If $k, \ell \in J^+$ and $a_{mn} \geq 0$ for $m, n \in J^+$, then $\sum_{m,n=1}^{\infty} a_{mn}$ is convergent if and only if $\sum_{m=k,n=\ell}^{\infty} a_{mn}$ is convergent (in which case there is the equality of theorem 1.3.2).

Remark 1.3.14

The hypothesis of Theorem 1.3.2 essentially calls for $\mathcal{J}$-convergence, σ-convergence and regular-convergence.

For double series in general, Neder [26, 1942] proved the following:

Theorem 1.3.3

A necessary and sufficient condition that the double series $\sum a_{ij}$, together with all of its rows and columns, be (regularly) convergent is that the partial sums

$$s_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$$

have a development of the form

$$s_{mn} = s - \varphi_m - \psi_n + \chi_{mn} ,$$

where $\varphi_m \to 0$ as $m \to \infty$, $\psi_n \to 0$ as $n \to \infty$ and $\chi_{m,n} \to 0$ as $m + n \to \infty$.

Later in 1947, Ogieveckii [29, 1947] proved the following:

Theorem 1.3.4

If $\alpha_{ij}(x)$ is a double sequence of functions such that

$$\sum_{i,j=1}^{\infty} |\Delta_{ij}\, \alpha_{ij}(x)| < K(x) ,$$

$$\lim_{n \to \infty} \Delta_i\, \alpha_{in}(x) = 0 ,$$

$$\lim_{m \to \infty} \Delta_j\, \alpha_{mj}(x) = 0 ,$$

and

$$\lim_{m,n \to \infty} \alpha_{mn} = 0 ,$$

where

$$\Delta_{ij}\, \alpha_{ij} = \alpha_{ij} - \alpha_{i,j+1} - \alpha_{i+1,j} + \alpha_{i+1,j+1} ,$$

$$\Delta_i\, \alpha_{ij} = \alpha_{ij} - \alpha_{i+1,j} ,$$

$$\Delta_j\, \alpha_{ij} = \alpha_{ij} - \alpha_{i,j+1} ,$$

then the double series $\sum \alpha_{ij}(x)\, u_{ij}(x)$ converges whenever

$\sum u_{ij}(x)$ has bounded partial sums.

Remark 1.3.15

For cases in which the differences are non-negative, the hypotheses are simplified and strengthened (by requiring uniformity of limits) to give uniform Pringsheim-convergence of $\sum \alpha_{ij}(x)\ u_{ij}(x)$ whenever $\sum u_{ij}(x)$ has uniformly bounded partial sums.

4. Tests for Convergence of Double Series (*)

The following are some of the tests for convergence of simple series which are also applicable to double series.

Theorem 1.4.1(a)

(Comparison test for convergence). If $|a_{mn}| \leq b_{mn}$ for $m, n \in J^+$ and $\sum\limits_{m,n=1}^{\infty} b_{mn}$ is convergent, then

$\sum\limits_{m,n=1}^{\infty} a_{mn}$ is absolutely convergent.

(*) See Livingston (1).

Theorem 1.4.1(b)

(Comparison test for divergence). If $0 \leq b_{mn} \leq |a_{mn}|$ for $m, n \in J^+$ and $\sum_{m,n=1}^{\infty} b_{mn}$ is divergent, then $\sum_{m,n=1}^{\infty} a_{mn}$ is not absolutely convergent.

Theorem 1.4.2

Let $a_{mn} = a'_{mn} + \sqrt{-1}\, a''_{mn}$, where a'_{mn} and a''_{mn} are real. Then $\sum_{m,n=1}^{\infty} a_{mn}$ converges if and only if $\sum_{m,n=1}^{\infty} a'_{mn}$ and $\sum_{m,n=1}^{\infty} a''_{mn}$ converge; and in this case

$$\sum_{m,n=1}^{\infty} a_{mn} = \sum_{m,n=1}^{\infty} a'_{mn} + \sqrt{-1} \sum_{m,n=1}^{\infty} a''_{mn} .$$

Theorem 1.4.3

(Ratio test). Suppose that $a_{mn} > 0$ for $m, n \in J^+$.

(i) If $\limsup_{m,n \to \infty} \left| \frac{a_{m+1,n}}{a_{mn}} \right| < 1$ and

$$\limsup_{m,n \to \infty} \left| \frac{a_{m,n+1}}{a_{mn}} \right| < 1 \quad , \quad \text{then}$$

$\sum_{m,n=1}^{\infty} a_{mn}$ is absolutely convergent.

(ii) If $\lim\limits_{m,n \to \infty} \inf \left| a_{m+1,n} \middle/ a_{mn} \right| > 1$ or

$$\lim_{m,n \to \infty} \inf \left| a_{m,n+1} \middle/ a_{mn} \right| > 1 \ , \text{ then}$$

$\sum\limits_{m,n=1}^{\infty} a_{mn}$ is not absolutely convergent.

Theorem 1.4.4

(Root test). Let $r = \lim\limits_{m,n \to \infty} \sup \sqrt[mn]{|a_{mn}|}$.

(i) If $r < 1$, then $\sum\limits_{m,n=1}^{\infty} a_{mn}$ is absolutely convergent.

(ii) If $r > 1$, then $\sum\limits_{m,n=1}^{\infty} a_{mn}$ is not absolutely convergent.

Theorem 1.4.5

(Integral test) [4, 1961]. Suppose $f(x, y)$ has the following properties:

(i) It is locally integrable and positive in $0 \le x < \infty$, $0 \le y < \infty$;

(ii) If $x < x'$ and $y < y'$, then

$$f(x', y') \le f(x, y) .$$

Then the double series $\sum\limits_{m=0}^{\infty} \sum\limits_{n=0}^{\infty} f(m, n)$ converges, if and

only if the three integrals $\int_0^\infty f(x, 0)\, dx$, $\int_0^\infty f(0, y)\, dy$

and $\int_0^\infty f(x, y)\, dx\, dy$ converge.

Example 1.4.1

The series

$$\sum_{m,n=1}^{\infty} 1 \Big/ (m + n)^p \quad , \quad p \text{ fixed, is convergent}$$

if $p > 2$ and divergent if $p \leq 2$.

Example 1.4.2

The series

$$\sum_{m,n=1}^{\infty} 1 \Big/ (m^p + n^q) \quad , \quad p \text{ and } q \text{ fixed, is}$$

convergent if $p, q > 2$ and divergent if $p \leq 2$ or $q \leq 2$.

Theorem 1.4.6

If $\sum_{m,n=1}^{\infty} a_{mn}$ is absolutely convergent to α , then each rearrangement of $\sum_{m,n=1}^{\infty} a_{mn}$ is absolutely convergent to α .

5. Convergence of certain Types of Double Series

A. Double power series

Let $\sum c_{st} x^s y^t$ $(x, y, c_{st} > 0)$ be a double power series. It is our aim to determine the region R of correlated variables (x, y) such that the series is convergent if $(x, y) \in R$ and divergent when $(x, y) \in \tilde{R}$.

Lembaire [5] in 1892 gave a formula that for a fixed value of $k = y/x$, the series is convergent for $x < r_1$, divergent for $x > r_1$ if

$$\frac{1}{r_1} = \overline{\lim} \sqrt[n]{(c_{st} k^t)} \quad , \quad (n = s + t).$$

Daniell [10, 1940] formulated a general test of the same type but expressed it differently:

Theorem 1.5.1

(Ratio test). Let

$$f(p) = \lim_{\substack{m \to \infty \\ \epsilon \to 0}} \sup_{\substack{s+t>m \\ |s/(s+t)-p| \leq \epsilon}} (s + t)^{-1} \log c_{st} \quad ,$$

where $0 \leq p \leq 1$. Then the series $\sum c_{st} x^s y^t$ is convergent

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(5) Lembaire, Bull. des Sci. Math. 20(1896), 286; Bromwich, Infinite series (1908), 504.

at (x, y) if for all p $(0 \leq p \leq 1)$,

$$p \log x + (1 - p) \log y + f(p) < 0 \quad ;$$

the series is divergent at (x, y) if for some p $(0 \leq p \leq 1)$

$$p \log x + (1 - p) \log y + f(p) > 0 \quad .$$

<u>Remark 1.5.1</u>

The above test is applicable to any series of finite multiplicity.

<u>Theorem 1.5.2</u>

Let the double power series $\sum c_{st} x^s y^t$ σ-converge for $x = x_o$, $y = y_o$. Then it converges absolutely for every (x, y) for which $|x| < |x_o|$, $|y| < |y_o|$, and converges uniformly in every closed region therein, thus representing an analytic function of the variables x, y in $|x| < |x_o|$, $|y| < |y_o|$.

<u>Corollary</u>

In Theorem 1.5.2 the hypothesis that there is σ-convergence for $x = x_o$, $y = y_o$ can be replaced by the weaker condition that the set of numbers $\{c_{st} x_o^s y_o^t\}$ is bounded.

Remark 1.5.2

Theorem 1.5.2 was proved first by Sheffer [33, 1945] and later on by Hitotumatu [14, 1952] independently. In addition, Hitotumatu showed the following special case.

Example 1.5.1

The power series with

$$a_{mn} = \begin{cases} 4 & m = n = 0 , \\ 1 & m = 0,\ n = 1 \text{ or } m = 1,\ n = 0, \\ 2 & m = 0,\ n \geq 2 \text{ or } m \geq 2,\ n = 0 \\ -2 & m = n = 1 \\ -1 & m = 1,\ n \geq 2 \text{ or } m \geq 2,\ n = 1, \\ 0 & m,n \geq 2; \end{cases}$$

i.e.

$$\sum_{m,n=0}^{\infty} a^{mn} x^m y^n = (2 - y) \sum_{m=0}^{\infty} x^m + \ldots + (2 - x) \sum_{n=0}^{\infty} y^n$$

is Pringsheim-convergent (but not A-convergent) at $x = 2$, $y = 2$ yet its absolute convergence region is not $|x| < 2$, $|y| < 2$, but is $|x| < 1$, $|y| < 1$.

By considering $x = y$, $\sum c_{st} x^s \cdot y^t$ is reduced to a multiple power series in one variable $\sum c_{st} u^{s+t}$. A theorem analogous to Theorem 1.5.2 is obtained for the new series.

Theorem 1.5.3

If the multiple power series $\sum c_{st} u^{s+t}$ in the variable u σ-converges for $u = u_o$, (or if the set of numbers $\{c_{st} u_o^{s+t}\}$ is bounded), then the series converges absolutely for all u in $|u| < |u_o|$, and σ-converges uniformly in every closed region therein, thus representing an analytic function of u in $|u| < |u_o|$.

Definition 1.5.1

The partial sum $C_n(u)$ is said to be a circular sum for the series $\sum a_{st} u^{s+t}$ if

$$C_n(u) = \sum_{\nu=0}^{n} b_\nu u^\nu$$

where $b_\nu = \sum a_{st}$, $(s + t = \nu)$.

Remark 1.5.3

Theorems 1.5.2 and 1.5.3 are false in the sense of Pringsheim-convergence.

Example 1.5.2

Consider the double power series

$$\begin{aligned} &0! + (1!)y + (2!)y^2 + (3!)y^3 + \dots + (n!)y^n + \dots \\ &-(0!)x - (1!)xy - (2!)xy^2 - (3!)xy^3 - \dots -(n!)xy^n - \dots \end{aligned}$$

with all other terms zero. By the rectangular sum definition, this series converges for $x = y = 1$, for $x = 1$, y arbitrary and for x arbitrary $y = 0$; but no other values.

Remark 1.5.4

Theorem 1.5.2 holds for Pringsheim-convergence if the series $\sum a_{st} x^s y^t$ converges absolutely for $x < x_o$, $y < y_o$.

Let

$$\zeta = \limsup |a_{st}|^{1/s+t} \quad , \quad (s + t \to \infty) \quad .$$

Parallel to Theorem 1.5.1, we have the following:

Theorem 1.5.4

The radius of convergence r of the series $\sum a_{st} u^{s+t}$ is given by $r = 1/\zeta$.

Theorem 1.5.5

The series $\sum a_{st} x^s y^t$ converges absolutely for all x, y for which $|x| < 1/\zeta$, $|y| < 1/\zeta$ and diverges for all x, y for which $|x| > 1/\zeta$, $|y| > 1/\zeta$.

Remark 1.5.5

The radius of convergence of $\sum a_{st} u^{s+t}$ by the circular sum definition is at least as great as r .

Example 1.5.3

If $\sum a_{st} u^{s+t}$ is the series whose first two rows are

$$1 + u + u^2 + \ldots + u^n + \ldots$$

$$1 - u - u^2 - \ldots - u^n - \ldots$$

while all other terms are 0 , then the circular sum definition gives the radius of convergence as 1 , which is also the value of r .

Example 1.5.4

If $\sum a_{st} u^{s+t}$ is the series having

$$1 + u + u^2 + \ldots + u^n + \ldots$$

as its first row and

$$1 - u - u^2 - \ldots - u^n - \ldots$$

as its first column, while all other terms are 0 , then by the circular sum definition, the series converges to the sum 1 for all n whereas $r = 1$.

Theorem 1.5.6 [9, 1946]

Let $\sum a_{st}$ be a double series with partial sums s_{mn} convergent to s . Let $s_{mn}/m \to 0$ for each n , $s_{mn}/n \to 0$

for each m and $s_{mn}/mn \to 0$ as $m, n \to \infty$. Let $\lambda > 0$. Then the double power series in

$$f(x, y) = \sum_{s,t=0}^{\infty} a_{st}\, x^s\, y^t$$

converges absolutely when $|x| < 1$, $|y| < 1$, and defines a function $f(x, y)$ such that

$$f(x, y) \to s \quad \text{as} \quad x \to 1 \ , \ y \to 1$$

subject to the restrictions $0 < x < 1$, $0 < y < 1$ and $\lambda^{-1} < (1 - x)/(1 - y) < \lambda$.

As another characterization for the convergence of a double power series, we have the following theorem due to Leja [19, 1951].

<u>Theorem 1.5.7</u>

If a diagonal series $\sum_{n=0}^{\infty} p_n(x, y)$, where

$$p_n(x, y) = \sum_{m=0}^{n} a_{m,n-m}\, x^m\, y^{n-m} \ ,$$

is absolutely convergent on the half-circle $x = r \cos t$, $y = r \sin t$, $0 \leq t < \pi$, $r > 0$, then it is also absolutely convergent for $|x + iy| < r$, $|x - iy| < r$, and the corresponding double power series $\sum_{p,q} a_{p,q}\, x^p\, y^q$ is absolutely convergent for $|x| + |y| < r$.

A recent result on double power series was again due to Leja [20, 1965].

Definition 1.5.2

Let $\sum_{s,t=0}^{\infty} a_{st} x^s y^t$ be a double power series and

$$s_{mn}(x, y) = \sum_{s=0}^{m} \sum_{t=0}^{n} a_{st} x^s y^t$$

its partial sum. The series is said to be almost bounded at the point (x, y) if there exists a number $M > 0$ and a $p \in J^+$ such that

$$|s_{mn}(x, y)| \leq M \quad , \quad \text{for} \quad m \geq p \quad \text{and} \quad n \geq p \quad .$$

Theorem 1.5.8

If the double power series $\sum_{s,t=0}^{\infty} a_{st} x^s y^t$ is convergent (or almost bounded) at the point (x_o, y_o) and at the points of two sequences $P_k(\xi_k, \eta)$, $Q_k(\xi, \eta_k)$, $k = 1, 2, 3, \ldots$ lying in the region

$$R = \{|x| < |x_o| \ , \ |y| < |y_o|\} \ ,$$

then the series is absolutely convergent in the region

$\{|x| < |\xi| \ , \ |y| < |\eta|\}$.

B. Cauchy product of double series

Definition 1.5.3

Let

$$A = \sum a_{ij} \quad ; \quad B = \sum b_{ij}$$

be two double series. The Cauchy product series derived from these two series is defined to be the series

$$C = \sum c_{ij} \ ,$$

where

$$c_{mn} = \sum a_{ij}\, b_{st} \ ,$$

the sum of which being over all indices i, j, s, t for which, simultaneously,

$$i + s = m \ ; \ j + t = n \ .$$

Definition 1.5.4

The series $\sum c_{ij}$ is said to possess the Cauchy product property if it converges and to the sum $C = AB$.

For simply-infinite series, Mertens proved that if both given series converge and one of them converges absolutely, then the Cauchy product property holds. But Mertens' assertion does not go over unrestrictedly to multiple series that are Pringsheim-convergent as can be seen from the following:

Example 1.5.5

Let $\sum a_{ij}$, $\sum b_{ij}$ be the following double series: $\sum a_{ij}$ is the absolutely convergent series whose first column has the elements

$$1 \ , \ \frac{1}{2} \ , \ \frac{1}{2^2} \ , \ \frac{1}{2^3} \ , \ \ldots$$

while all other terms are zero; $\sum b_{ij}$ is the Pringsheim-convergent series whose first two rows are

$$0! + 1! + \ldots + n! + \ldots$$
$$-(0!) - (1!) - \ldots - (n!) - \ldots$$

all other terms being zero. We see that in the series $\sum c_{ij}$,

$$c_{oj} = j! \ ; \ c_{ij} = -j!/2^i \ , \quad (i > 0) \ ,$$

and that

$$R^c_{pq} = \frac{1}{2^p}(0! + 1! + \ldots + q!) \ , \tag{1.5.1}$$

where R^c_{pq} designates the $(p, q)^{th}$ rectangular partial sum in the C-series.

$$R^c_{pq} = \sum c_{ij} \ , \quad (0 \leq i \leq p \ , \ 0 \leq j \leq q) \ . \tag{1.5.2}$$

Equation (1.5.1) shows that R^c_{pq} does not have a limit as $p, q \to \infty$; so $\sum c_{ij}$ does not converge.

Amerio [3, 1943] proved the following theorem on Cauchy productsof double series.

Theorem 1.5.9

If $\sum_{i,j=0}^{\infty} a_{ij}$ converges by row to A i.e.

$\sum_{i,j=0}^{\infty} a_{ij} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} = A$; and if $\sum b_{ij}$ converges to B and converges absolutely, then the Cauchy product series $\sum_{i,j=0}^{\infty} c_{ij}$ converges by row to $A\,B$, i.e.

$$\sum_{i,j=0}^{\infty} c_{ij} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} = C = AB \ .$$

Three years later, Sheffer [34, 1946] proved a more general result.

Theorem 1.5.10

Let $A = \sum a_{ij}$ be absolutely convergent, $B = \sum b_{ij}$ is Pringsheim-convergent. If the set of rectangular partial sums R^B of $\sum b_{ij}$ is bounded, then $\sum c_{ij}$ converges to the sum $C = AB$.

Hardy [2, 1911] proved the following theorem for doubly positive double series.

Theorem 1.1.9

(i) $\sum_{i,j=0}^{\infty} a_{ij}$ converges to the sum s (i.e.,

$\sum_{i,j=0}^{m,n} a_{ij} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij} \to s$), and if $\sum_{j} a_{ij}$ converges to

s_i and converges absolutely, then the double repeated series

$\sum_{i,j=0}^{\infty} a_{ij}$ converges to the sum s (i.e.,

$$\sum_{i,j=0}^{m,n} a_{ij} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij} \to s \text{ as } m, n \to \infty).$$

Three years later, Moricz [6, 1914] proved a more general result.

Theorem 1.1.10

Let $A = \sum a_{ij}$ be absolutely convergent, $B = \sum b_{ij}$

be Pringsheim-convergent. If the sum of the convolution series,

with c_{ij} of $\sum c_{ij}$ is bounded, then $\sum c_{ij}$ converges

to the sum $C = AB$.

He was also able to assert the following:

Theorem 1.5.11

Let $B = \sum b_{ij}$ be convergent. If the set $\{R^B_{pq}\}$ is unbounded, then there exists an absolutely convergent series $\sum a_{ij}$ for which the Cauchy product property fails to hold; in fact, for which series $\sum c_{ij}$ does not converge.

Combining Theorems 1.5.10 and 1.5.11, we have:

Theorem 1.5.12

Let $B = \sum b_{ij}$ be Pringsheim-convergent. In order that the Cauchy product series $\sum c_{ij}$ converge to the sum $C = AB$ for every absolutely convergent series $A = \sum a_{ij}$, it is necessary and sufficient that the set $\{R^B_{pq}\}$ be bounded.

Remark 1.5.6

Since the property of being σ-convergent carries with it the boundedness of the set of all σ-sums, hence following the method of Theorem 1.5.10, Mertens' theorem for simply-infinite series can be carried over to doubly-infinite series.

Theorem 1.5.13

If $\sum a_{ij}$, $\sum b_{ij}$ are σ-convergent, one of them being absolutely convergent, then the Cauchy product property holds: series $\sum c_{ij}$ is σ-convergent to the sum $C = AB$.

C. Dirichlet product of double series [22, 1928]

Given the double series $\sum_{m,n=1}^{\infty} a_{mn}$ and $\sum_{m,n=1}^{\infty} b_{mn}$. Form the Dirichlet double series

$$\sum_{m,n=1}^{\infty} a_{mn} e^{-\lambda_m' s - \mu_n' t} \quad \text{and} \quad \sum_{m,n=1}^{\infty} b_{mn} e^{-\lambda_m'' s - \mu_n'' t} .$$

Let λ_i be the ascending sequence formed by all the values of $\lambda_p' + \lambda_q''$, and μ_j the ascending sequence formed by all the values of $\mu_{p'}' + \mu_{q'}''$. If

$$\lambda_{p_1}' + \lambda_{q_1}'' = \lambda_{p_2}' + \lambda_{q_2}'' ,$$

so that the two values of λ are the same, the order of these two values is indifferent. The same thing is true for two equal values of μ .

Definition 1.5.5

The series $\sum_{i,j=1}^{\infty} c_{ij}$, where

$$c_{ij} = \sum_{\lambda'_m+\lambda''_k=\lambda_i} \sum_{\mu'_n+\mu''_\ell=\mu_j} a_{mn} b_{k\ell} ,$$

is called the Dirichlet product of the double series $\sum_{m,n=1}^{\infty} a_{mn}$, $\sum_{m,n=1}^{\infty} b_{mn}$ of type $(\lambda' , \mu' ; \lambda'' , \mu'')$.

Remark 1.5.7

Since the Cauchy product series as defined above is a special case of the Dirichlet product series, it follows that the following theorems are also true for the Cauchy product of double series.

Theorem 1.5.14

If $\sum_{m,n=1}^{\infty} a_{mn}$ is absolutely convergent to sum A , and $\sum_{m,n=1}^{\infty} b_m$ is Pringsheim-convergent to sum B and bounded, then $\sum_{i,j=1}^{\infty} c_{ij}$, the Dirichlet product of the type $(\lambda' , \mu' ; \lambda'' , \mu'')$, is Pringsheim-convergent to sum C and bounded, and $C = AB$.

Theorem 1.5.15

If $\sum_{m,n=1}^{\infty} a_{mn}$, $\sum_{m,n=1}^{\infty} b_{mn}$ and $\sum_{m,n=1}^{\infty} c_{mn}$ are Pringsheim-convergent and bounded, where $\sum_{m,n=1}^{\infty} c_{mn}$ is the Dirichlet product, then $AB = C$.

Theorem 1.5.16

If $\sum_{m,n=1}^{\infty} a_{mn}$ and $\sum_{m,n=1}^{\infty} b_{mn}$ are absolutely convergent with sums A and B respectively, then the Dirichlet product series of type $(\lambda' , \mu' ; \lambda'' , \mu'')$ is absolutely convergent with sum AB .

D. Harmonic double series

Let the series $\sum_{p,q=1}^{\infty} p^{-r} q^{-s} (p + q)^{-t}$ be termed a harmonic double series and write

$$(r, s, t) = \sum_{p,q=1}^{\infty} p^{-r} q^{-s} (p + q)^{-t} ,$$

then we have the following theorem due to Tornheim [35, 1950].

<u>Theorem 1.5.17</u>

The harmonic double series $\sum_{p,q=1}^{\infty} p^{-r} q^{-s} (p+q)^{-t}$ is finite if and only if

$$r + t > 1 \ , \quad s + t > 1 \quad \text{and} \quad r + s + t > 2 \ .$$

E. <u>Alternating double series</u>

Let $\sum_{i,j=1}^{\infty} a_{ij}$ be a double series. We call the partial sums $D_k = \sum_{i+j=k} a_{ij}$ the diagonal of the series.

<u>Definition 1.5.6</u>

A series is said to be diagonally summable if $\sum_{n=2}^{\infty} D_n$ converges.

<u>Definition 1.5.7</u>

A double series is said to be alternating if each row and each column is an alternating single series.

<u>Definition 1.5.8</u>

The alternating series $\sum_{i,j=1}^{\infty} a_{ij}$ is said to be monotonic if

$$|a_{ij}| \leq |a_{mn}| \ , \quad \text{for} \ i \geq m \ , \ j \geq n \ .$$

The following theorem was established by Meyer [23, 1953] for the σ-convergence of an alternating series.

Theorem 1.5.18

A necessary and sufficient condition for the σ-convergence of a harmonic alternating double series

$$\sum_{i,j=1}^{\infty} a_{ij}$$ is that it is diagonally summable.

The following example provides an interesting comparison of the definitions of σ-convergence and regular-convergence.

Example 1.5.6 [23, 1953]

Consider the series

$$\sum_{i,j=1}^{\infty} (-1)^{i+j} (i + j - 1)^{-n} .$$

Since

$$\int_1^{\infty}\int_1^{\infty} (x + y - 1)^{-n} \, dx \, dy$$

converges for $n > 2$, the above series is absolutely Pringsheim-convergent for $n > 2$ (see [8, 1926]). Pringsheim has shown [32, 1916] that a monotonic alternating double series is Pringsheim-convergent if

$$|a_{ij} + a_{i+1,j}| \geq |a_{i,j+1} + a_{i+1,j+1}|$$

for all i and j. The series satisfies this condition for $n > 0$, since $x^{-n} - (x + 1)^{-n}$ is a decreasing function of x for $x \geq 1$, $n > 0$. Hence the above double series is regularly convergent if $n > 0$, and it diverges for $n \leq 0$.

<u>Theorem 1.5.18</u>

A necessary and sufficient condition for the P-convergence of a summable alternating double series

$\sum_{i,j=1}^{\infty} a_{ij}$ is that it is regularly convergent.

The following example provides an interesting comparison of the strengths of P-convergence and Regular-convergence.

<u>Example 1.5.19</u> (RK, 1977)

Consider the series

$$\sum_{i,j=1}^{\infty} (-1)^{i+j} (i + j - 1)^{-\alpha} .$$

Since

$$\int_1^{\infty}\int_1^{\infty} (x + y - 1)^{-\alpha}\, dx\, dy$$

converges for $\alpha > 2$, the above series is absolutely Pringsheim-convergent for $\alpha > 2$ [see [9, 1963]]. Kronsbein has shown [29, 1965] that a monotonic alternating double series is regularly-convergent if

$$\{a_{ij} - a_{i+1,j}\} \ge \{a_{i,j+1} - a_{i+1,j+1}\}$$

for all i and j. The series satisfies this condition for $\alpha > 0$, since $x^{-\alpha} - (x+1)^{-\alpha}$ is a decreasing function of x for $x \geq 1$, $\alpha > 0$. Hence the above double series is regularly convergent if $\alpha > 0$, and it diverges for $\alpha \leq 0$.

Setting $n = 1$ and then summing the series by diagonals, we obtain

$$1 - 1 + 1 - 1 + \dots \quad .$$

Therefore this regularly convergent series is not σ-convergent. The above theorem shows that the series is σ-convergent if and only if $n > 1$.

CHAPTER II

SUMMABILITY OF DOUBLE SEQUENCES AND SERIES (I)

(GENERAL METHODS)

1. Types of Convergence

A double series $\sum_{i,j=0}^{\infty} a_{ij}$ may be classified according to the behaviour of the double sequences of its partial sums

$$s_{mn} = \sum_{i,j=0}^{m,n} a_{ij}$$

as follows:

A. Principal types

Definition 2.1.1

The sequence $\{s_{mn}\}$ is said to be

(i) ultimately bounded (abbreviated (ub)) if there exists a number Q such that s_{mn} is bounded for all $m , n > Q$;

(ii) bounded (b) if in the preceding case Q can be taken to be zero.

Definition 2.1.2

The sequence $\{s_{mn}\}$ is said to be

(i) convergent (c) if $\lim\limits_{m,n \to \infty} s_{mn} = s$ (read "Principal limit") exists and is finite;

(ii) boundedly convergent $((b) \cap (c))$ if (c) and if s_{mn} is bounded for all m, n;

(iii) ultimately regularly convergent (u r c) if (c), and if there exists a number Q such that $\lim\limits_{m \to \infty} s_{mn} = s_{.n}$ and $\lim\limits_{n \to \infty} s_{mn} = s_{m.}$ both exist for all $n > Q$ and for all $m > Q$, respectively;

(iv) regularly convergent (rc) if in the preceding case Q can be taken to be zero;

(v) boundedly ultimately regularly convergent $((b) \cap (u\,r\,c))$ if both (b) and (u r c).

Definition 2.1.3

The sequence $\{s_{mn}\}$ is said to be perfectly convergent (pc) if it is (rc) and $s_{m.} = s_{.n}$ for m, $n = 0, 1, 2, \ldots$.

Remark 2.1.1

Throughout this chapter finiteness is assumed whenever the limit exists and convergence is understood to be in the sense of Pringsheim.

Remark 2.1.2

Every regularly convergent sequence is bounded.

Definition 2.1.4

Denote by E_k the set of all pairs (m, n) of indices such that $k^{-1} \leq (m+1)(n+1)^{-1} \leq k$; the sequence $\{s_{mn}\}$ is called restrictedly convergent to s if, given any $\epsilon > 0$ and k, there is a K such that $m, n > K$ together with $(m, n) \in E_k$ implies

$$|s_{mn} - s| < \epsilon \quad ;$$

s is then called the limit, in the restricted sense, of the sequence $\{s_{mn}\}$ and is denoted by $[\lim_{m,n \to \infty}] s_{mn} = s$.

B. Auxiliary types

Definition 2.1.5 [25, 1948]

The sequence $\{s_{mn}\}$ is said to be absolutely convergent (ac), if there exists a number P such that $\sum_{i,j=1}^{\infty} |s_{ij} - s_{i-1,j} - s_{i,j-1} + s_{i-1,j-1}| = P$.

Definition 2.1.6

The sequence $\{s_{mn}\}$ is said to be

(i) a convergent null sequence (c_o) if it is (c) and $s = 0$;

(ii) a convergent row null sequence $((c) \cap (r_o))$ if it is convergent and $s_{mn} = 0$ for all $n > N$, $m = 0, 1, 2, \ldots$.

Corresponding to the types of convergent sequences given in definition 2.1.2 there are several special types. We list only their names, and their definitions, being analogous to those in definition 2.1.2, are omitted.

Absolutely convergent null sequences

(i) absolutely convergent null sequences (ac_o) .

(ii) absolutely ultimately regularly convergent null sequences $(a\ u\ r\ c_o)$.

(iii) absolutely regularly convergent null sequences $(a\ r\ c_o)$.

Null sequences

(i) convergent null sequences (c_o) .

(ii) boundedly convergent null sequences $((b) \cap (c_o))$.

(iii) ultimately regularly convergent null sequences $(u\ r\ c_o)$.

(iv) regularly convergent null sequences $(r\ c_o)$.

(v) boundedly ultimately regularly convergent null sequences $((b) \cap (u\ r\ c_o))$.

Row null sequences

(i) ultimately regularly convergent row null sequences $((u\ r\ c) \cap (r_o))$.

(ii) regularly convergent ultimately row null sequences $((r\ c) \cap (u\ r_o))$.

(iii) boundedly ultimately regularly convergent row null sequences $((b) \cap (u\ r\ c) \cap (r_o))$.

(iv) regularly convergent row null sequences $((r\ c) \cap (r_o))$.

2. Types of Transformations

For a given double sequence of partial sums

$$s_{mn} = \sum_{i,j=0}^{m,n} x_{ij}$$

of the double series $\sum_{i,j=0}^{\infty} x_{ij}$, there are two most general methods of linear transformations, one by a triangular matrix and the other by a square matrix.

Define a new sequence by the relation

$$\sigma_{mn} = \sum_{i,j=1}^{m,n} a_{mnij}\, s_{ij} \ .$$

We shall call this transformation and its matrix $A : (a_{mnij})$ of the type T ; here $i \leq m$, $j \leq n$.

We may write

$$\sigma_{mn} = \sum_{i,j=1}^{\infty} a_{mnij}\, s_{ij}$$

provided σ_{mn} has a meaning, and call this transform and its matrix $A : (a_{mnij})$ of the type S ; here i and j take on all positive integral values.

Any transformation of type T may be considered as a special case of type S ; for by adding the elements

$$a_{mnij} = 0 \qquad m < i \ , \ n < j \ , \quad \text{all } m \text{ and } n \ ,$$

$$a_{mnij} = 0 \qquad 1 \leq i \leq m \ , \ n < j \ , \text{all } m \text{ and } n \ ,$$

$$a_{mnij} = 0 \qquad m < i \ , \ 1 \leq j \leq n \ , \text{all } m \text{ and } n \ ,$$

to any matrix of type T we obtain a matrix of type S such that the resulting transformation is identical with the original one. If for either transformation σ_{mn} possesses a limit, then the limit is called the generalized value of the sequence s_{mn} by the transformation [30, 1926].

Remark 2.2.1

In the subsequent sections we use "method A" and "type S" interchangeably.

Definition 2.2.1

The sequence $\{s_{ij}\}$ is said to be A-summable (or to be summable by the method A) to σ if the sequence of transformations $\{\sigma_{mn}\}$ converges to σ .

Notation

The number σ will be denoted by $A - \lim\limits_{i,j \to \infty} s_{ij}$ or simply σ .

Definition 2.2.2

The sequence $\{s_{ij}\}$ is said to be regularly A-summable to σ if the sequence $\{\sigma_{mn}\}$ converges regularly to σ and we write

$$\lim_{m \to \infty} \sigma_{mn} = \sigma_{.n} \quad ; \quad \lim_{n \to \infty} \sigma_{mn} = \sigma_{m.} \quad .$$

Definition 2.2.3

A regular transformation of a regularly convergent sequence into a regularly convergent sequence is said to be completely regular (or completely permanent) if it is also regular by row and by column, i.e.

$$\sigma_{m.} = \lim_{n \to \infty} \sigma_{mn} = s_{m.} \ ; \ \sigma_{.n} = \lim_{m \to \infty} \sigma_{mn} = s_{.n} \ ,$$

$m, n = 0, 1, 2, \ldots$.

<u>Definition 2.2.4</u>

The method A is said to be completely conservative for a certain space, not containing a divergent sequence, if all the double sequences belonging to that space are transformed into sequences of the same space.

<u>Definition 2.2.5</u>

The sequence $\{s_{ij}\}$ is said to be perfectly A-summable if it is regularly A-summable and $\sigma_{m.} = \sigma_{.n}$ for $m, n = 0, 1, 2, \ldots$, i.e. if the sequence of transforms converges perfectly.

<u>Definition 2.2.6</u>

The method A is said to fulfill the condition (r_o) if it transforms every sequence $\{s_{ij}\}$ regularly convergent to zero into a sequence regularly convergent in such a manner that

$$\sigma_{m.} = s_{m.} \; ; \; \sigma_{.n} = s_{.n} \quad \text{for } m, n = 0, 1, 2, \ldots .$$

In this case, of course, $\sigma = 0$.

<u>Definition 2.2.7</u>

The sequence $\{s_{ij}\}$ is said to be restrictedly A-summable to σ if $[\lim_{m,n \to \infty}] \sigma_{mn} = \sigma$ exists. σ is also denoted by $A - [\lim_{m,n \to \infty}] s_{mn}$.

Definition 2.2.8

Two methods A and B are said to be consistent if $s_{ij} \to s\ (A)$ and $s_{ij} \to s'(B)$ imply $s = s'$, i.e. if they cannot sum the series into different sums.

3. Questions on Summability of Double Sequences

The problems concerning the summability of double sequences can be summarized by the following questions:

(i) If $\{s_{ij}\}$ is of a specified one of the types defined in § 1.A, under what conditions on $A : (a_{mnij})$ will $\{\sigma_{mn}\}$ be of a specified one of these types?

(ii) If $\{s_{ij}\}$ is of a specified one of the types defined by definition 2.1.2 , under what conditions will $\{\sigma_{mn}\}$ be of a specified one of these types, with $\sigma = s$?

(iii) If $\{s_{ij}\}$ is of a specified one of the types in definition 2.1.2 (involving regular convergence, either complete or deferred), under what conditions will $\{\sigma_{mn}\}$ be of a specified one of these types, with $\sigma_{.n} = s_{.n}$ for all n sufficiently large?

(iv) If $\{s_{ij}\}$ is regularly convergent, under what conditions will $\{\sigma_{mn}\}$ be regularly convergent with $\sigma_{.n} = s_{.n}$ for all n ?

(v) If $\{s_{ij}\}$ is regularly convergent, under what conditions will $\{\sigma_{mn}\}$ be completely permanent i.e. the transformation preserves not only the principal limit but also the row and column limits?

<u>Remark 2.3.1</u>

Since under the infinite matrix transformation σ_{mn} may not even exist for certain values of m and n, it becomes necessary to speak of the class of sequences, all of whose elements exist. Such sequences are called existent (abbreviated (e)).

<u>Notation</u>

The process of transforming will be indicated by T. "Necessary" will be abbreviated by N, "sufficient" by S, and "regularity", as applied to transformations by (reg). Thus "$S \quad T(c) \subset (r\ c) \cap (\text{reg})$" reads "a set of conditions sufficient that every convergent sequence will be transformed into a regularly convergent sequence with preservation of the principal limit".

4. <u>Earlier Results</u>

Attempts to answer the questions of §3 were made first by Kojima in 1922 [19, 1922]. He establishes the conditions necessary and sufficient for the transformations of type T to give the following results:

(i) $T(c_o) \subset (c_o)$

(ii) $T(c_o) \subset (c)$

(iii) $T(c) \subset (c)$

(iv) $T(c) \subset (c) \cap (reg)$

(v) $T((b) \cap (c)) \subset (c)$ (1)

(vi) $T(r\ c) \subset (c)$

(vii) $T(r\ c) \subset (r\ c)$.

In 1926 Robison [30, 1926] asserts, independently, conditions necessary and sufficient for the transformations of type T to give

(i) $T((b) \cap (c)) \subset (c) \cap (reg)$

(ii) $T((b) \cap (c)) \subset (b) \cap (c)$

(iii) $T((b) \cap (c)) \subset (c)$, with σ a function of s only,

(iv) $T(b) \subset (b) \cap (c)$,

and for the transformations of type S to give

(i) $T((b) \cap (c)) \subset (e) \cap (c) \cap (reg)$

(ii) $T((b) \cap (c)) \subset (e) \cap (b) \cap (c)$

(iii) $T((b) \cap (c)) \subset (e) \cap (c)$, with σ a function of s only,

(iv) $T(b) \subset (e) \cap (b) \cap (c)$.

---

(1) Kojima, On the theory of double sequences, Tohoku Math. J. 21(1922), p. 12, Theorem V. Obviously typographical errors in the list of conditions can be remedied by studying the context.

Leja [20, 1930] in 1930 gives the necessary and sufficient conditions for the transformations of both type T and type S for the following:

(i) $T((b) \cap (c)) \subset (c)$

(ii) $T(c) \subset (c)$

(iii) $T(c) \subset (c) \cap (reg)$.

Agnew [3, 1932] establishes the necessary and sufficient conditions for the transformations of type T that

(i) $T(c) \subset (ub) \cap (c) \cap (reg)$

(ii) $T(c) \subset (c) \cap (reg)$.

Hallenbach [13, 1933] establishes conditions necessary and sufficient for the transformations of type S to give

(i) $T(c) \subset (e) \cap (c)$

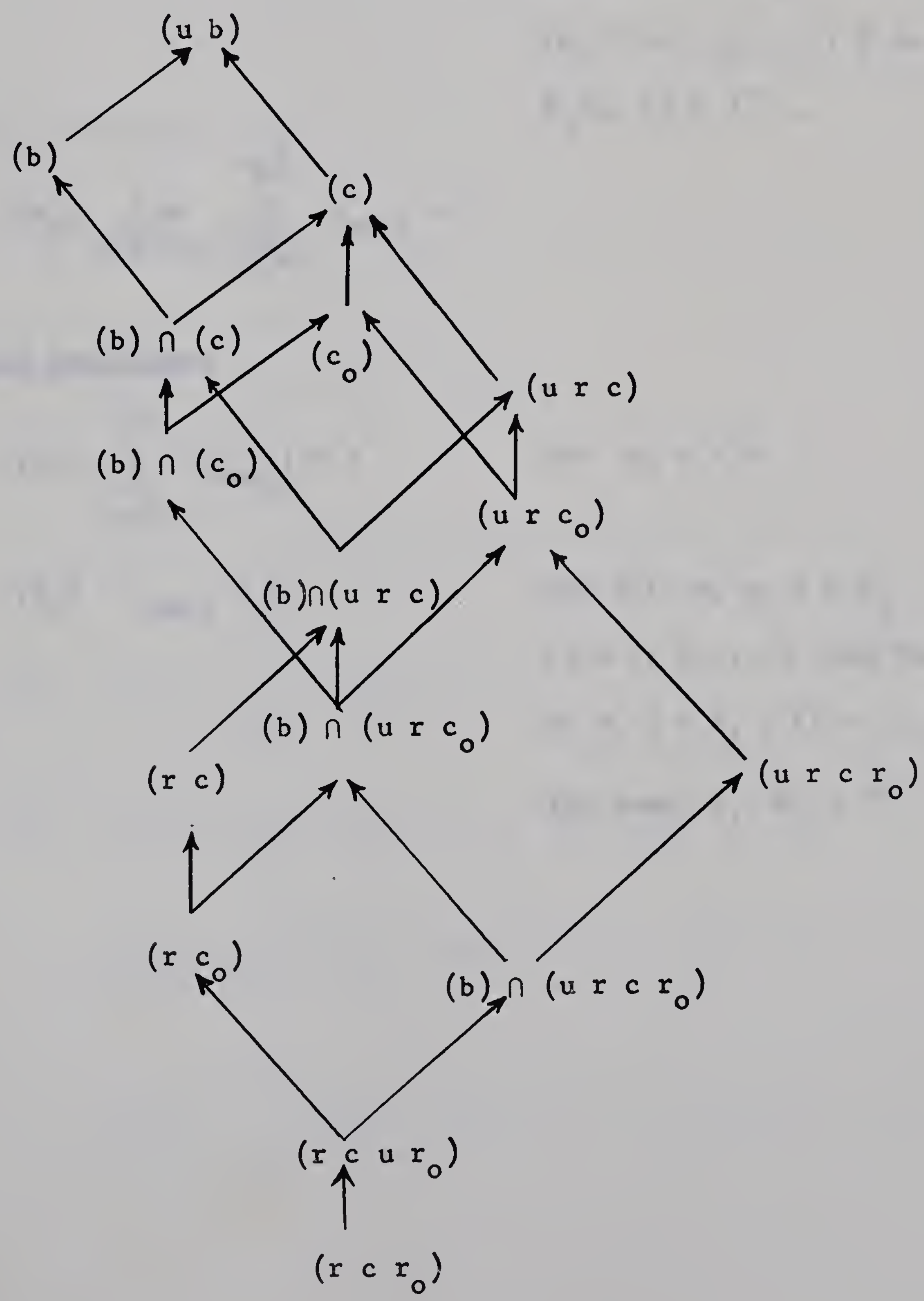
(ii) $T(b) \subset (b)$

(iii) $T((b) \cap (c)) \subset (b) \cap (c)$.

These results, while they are obtained independently by the respective authors, overlap in many cases. The most significant of them being the necessary and sufficient conditions for the transformations $T(c) \subset (c) \cap (reg)$ which was established first by Kojima and later by Leja, Adams [1, 1932] and Agnew. We shall state, in a later section, the theorem due to Kojima.

5. Schematic Relation of the Types of Sequences

We give a diagram designed by Hamilton showing the relation between the various types of sequences. Under the hypothesis that (rc) sequences are (e) , the several types of sequences to be considered are related as shown in the diagram, the arrow indicating implication of the quality at its head by that at its tail [14, 1936].



6. Conditions on the Matrix [14, 1936]

Existence conditions

(a_1) $\sum_{i,j=1}^{\infty} |a_{mnij}| < \infty$ $\quad (m, n = 1, 2, \ldots)$,

(a_2) $a_{mnij} = 0$ $\quad$ for $i > M_j\ (m, n)$, $(j = 1, 2, \ldots)$;

and for $j > M_i\ (m, n)$, $(i = 1, 2, \ldots)$;

$(m, n = 1, 2, \ldots)$; for some $M_i(m, n)$, $M_j(m, n) \in J^+$,

(a_3) $\lim_{m,n \to \infty} \sum_{i,j=1}^{m,n} a_{mnij} = 1$.

(ub) conditions

(b_1) $\sum_{i,j=1}^{\infty} |a_{mnij}| < A$ $\quad$ for $m, n > B$,

(b_2) $a_{mnij} = 0$ $\quad$ for all $m, n, i > M_j$, $(j = 1, 2, \ldots)$ and for all $m, n, j > M_i$, $(i = 1, 2, \ldots)$ for some $M_i, M_j \in J^+$.

(b) conditions

(c_1) $\sum_{i,j=1}^{\infty} |a_{mnij}| < A$ $(m, n = 1, 2, \ldots)$,

(c_2) $a_{mnij} = 0$ for $i > M_j$, $(j = 1, 2, \ldots)$

and for $j > M_i$, $(i = 1, 2, \ldots)$

for some M_i , $M_j \in J^+$,

$(m, n = 1, 2, \ldots)$.

(c) conditions

(d_1) $\lim_{m,n \to \infty} a_{mnij} = a_{..ij}$ $(i, j = 1, 2, \ldots)$,

(d_2) $\lim_{m,n \to \infty} \sum_{i=1}^{\infty} a_{mnij} = L_{...j}$,

$\lim_{m,n \to \infty} \sum_{j=1}^{\infty} a_{mnij} = L_{..i.}$ $(i, j = 1, 2, \ldots)$,

(d_3) $\lim_{m,n \to \infty} \sum_{i,j=1}^{\infty} a_{mnij} = L$,

(d_4) There exist numbers b_{ij} such that

$\lim_{m,n \to \infty} \sum_{i=1}^{\infty} |a_{mnij} - b_{ij}| = 0$ $(j = 1, 2, \ldots)$,

$\lim_{m,n \to \infty} \sum_{j=1}^{\infty} |a_{mnij} - b_{ij}| = 0$ $(i = 1, 2, \ldots)$,

(d_5) There exist numbers b_{ij} such that

$$\lim_{m,n \to \infty} \sum_{i,j=1}^{\infty} |a_{mnij} - b_{ij}| = 0 \quad .$$

<u>(c_o) conditions</u>

$(\bar{d}_1)$ (d_1) , with $a_{..ij} = 0$ $(i, j = 1, 2, \ldots)$,

$(\bar{d}_2)$ (d_2) , with $L_{..i.}$, $L_{...j} = 0$ $(i, j = 1, 2, \ldots)$,

$(\bar{d}_3)$ (d_3) , with $L = 0$,

$(\bar{d}_4)$ (d_4) , with $b_{ij} = 0$ $(i, j = 1, 2, \ldots)$,

$(\bar{d}_5)$ (d_5) , with $b_{ij} = 0$ $(i, j = 1, 2, \ldots)$.

<u>$(u\ r\ c)$ conditions</u>

(e_1) $\lim_{m \to \infty} a_{mnij} = a_{.nij}$ for $n > D$,

$\lim_{n \to \infty} a_{mnij} = a_{m.ij}$ for $m > D$ $(i, j = 1, 2, \ldots)$,

$(e_2)^*$ $\lim_{m \to \infty} \sum_{i=1}^{\infty} a_{mnij} = L_{.n.j}$, for $n > E_j$,

$\lim_{n \to \infty} \sum_{i=1}^{\infty} a_{mnij} = L_{m..j}$, for $m > E_j$,

$(j = 1, 2, \ldots)$,

$\lim_{m \to \infty} \sum_{j=1}^{\infty} a_{mnij} = L_{.ni.}$, for $n > E_i$,

$$\lim_{n \to \infty} \sum_{j=1}^{\infty} a_{mnij} = L_{m.i.} \quad \text{for} \quad m > E_i \qquad (i = 1, 2, \ldots),$$

(e$_2$)
$$\lim_{m \to \infty} \sum_{i=1}^{\infty} a_{mnij} = L_{.n.j}, \quad \text{for} \quad n > E,$$

$$\lim_{n \to \infty} \sum_{i=1}^{\infty} a_{mnij} = L_{m..j}, \quad \text{for} \quad m > E \qquad (j = 1, 2, \ldots),$$

$$\lim_{m \to \infty} \sum_{j=1}^{\infty} a_{mnij} = L_{.ni.}, \quad \text{for} \quad n > E,$$

$$\lim_{n \to \infty} \sum_{j=1}^{\infty} a_{mnij} = L_{m.i.}, \quad \text{for} \quad m > E \qquad (i = 1, 2, \ldots),$$

(e$_3$)
$$\lim_{m \to \infty} \sum_{i,j=1}^{\infty} a_{mnij} = L_{.n..}, \quad \text{for} \quad n > F,$$

$$\lim_{n \to \infty} \sum_{i,j=1}^{\infty} a_{mnij} = L_{m...}, \quad \text{for} \quad m > F.$$

(e$_4$)* There exist numbers b_{mij} such that

$$\lim_{n \to \infty} \sum_{i=1}^{\infty} |a_{mnij} - b_{mij}| = 0 \quad \text{for} \quad m > G_j \qquad (j = 1, 2, \ldots),$$

$$\lim_{n \to \infty} \sum_{j=1}^{\infty} |a_{mnij} - b_{mij}| = 0 \quad \text{for} \quad m > G_i \qquad (i = 1, 2, \ldots),$$

and there exist numbers b_{nij} such that

$$\lim_{m \to \infty} \sum_{i=1}^{\infty} |a_{mnij} - b_{nij}| = 0 \quad \text{for} \quad n > G_j \qquad (j = 1, 2, \ldots),$$

$$\lim_{m \to \infty} \sum_{j=1}^{\infty} |a_{mnij} - b_{nij}| = 0 \quad \text{for} \quad n > G_i \qquad (i = 1, 2, \ldots) \ .$$

(e_4) There exist numbers b_{mij} such that

$$\lim_{n \to \infty} \sum_{i=1}^{\infty} |a_{mnij} - b_{mij}| = 0 \quad \text{for} \quad m > G \qquad (j = 1, 2, \ldots)$$

$$\lim_{n \to \infty} \sum_{j=1}^{\infty} |a_{mnij} - b_{mij}| = 0 \quad \text{for} \quad m > G \qquad (i = 1, 2, \ldots),$$

and there exist numbers b_{nij} such that

$$\lim_{m \to \infty} \sum_{i=1}^{\infty} |a_{mnij} - b_{nij}| = 0 \qquad n > G \qquad (j = 1, 2, \ldots),$$

$$\lim_{m \to \infty} \sum_{j=1}^{\infty} |a_{mnij} - b_{nij}| = 0 \qquad n > G \qquad (i = 1, 2, \ldots) \ .$$

(e_5) There exist numbers b_{mij} such that

$$\lim_{n \to \infty} \sum_{i,j=1}^{\infty} |a_{mnij} - b_{mij}| = 0 \quad \text{for} \quad m > H$$

and numbers b_{nij} such that

$$\lim_{m \to \infty} \sum_{i,j=1}^{\infty} |a_{mnij} - b_{nij}| = 0 \quad \text{for} \quad n > H \ .$$

<u>$(u\ r\ c) \cap (r_o)$ conditions</u>

$(\bar{e}_1)$ (e_1) , with

$a_{.nij} = 0$ for $n > \bar{D}$,

$a_{m.ij} = 0$ for $m > \bar{D}$, $(i, j = 1, 2, \ldots)$,

$(e_2)^*$ $(e_2)^*$, with

$L_{.n.j} = 0$ for $n > \bar{E}_j$,

$L_{m..j} = 0$ for $m > \bar{E}_j$, $(j = 1, 2, \ldots)$,

$L_{.ni.} = 0$ for $n > \bar{E}_i$,

$L_{m.i.} = 0$ for $m > \bar{E}_i$, $(i = 1, 2, \ldots)$,

$(\bar{e}_2)$ (e_2) , with

$L_{.n.j} = 0$ for $n > \bar{E}$,

$L_{m..j} = 0$ for $m > \bar{E}$, $(j = 1, 2, \ldots)$,

$L_{.ni.} = 0$ for $n > \bar{E}$,

$L_{m.i.} = 0$ for $m > \bar{E}$, $(i = 1, 2, \ldots)$,

$(\bar{e}_3)$ (e_3) , with

$L_{.n..} = 0$ for $n > \bar{F}$,

$L_{m...} = 0$ for $m > \bar{F}$,

$(\bar{e}_4)^*$ $(e_4)^*$, with

$$b_{mij} = 0 \qquad \text{for } m > \bar{G}_i , \bar{G}_j ,$$

$$b_{nij} = 0 \qquad \text{for } n > \bar{G}_i , \bar{G}_j , \quad (i, j = 1, 2, \ldots) ,$$

$(\bar{e}_4)$ (e_4) , with

$$b_{mij} = 0 \qquad \text{for } m > \bar{G} ,$$

$$b_{nij} = 0 \qquad \text{for } n > \bar{G} , \qquad (i, j = 1, 2, \ldots) ,$$

$(\bar{e}_5)$ (e_5) , with

$$b_{mij} = 0 \qquad \text{for } m > \bar{H} ,$$

$$b_{nij} = 0 \qquad \text{for } n > \bar{H} , \qquad (i, j = 1, 2, \ldots) .$$

<u>(r c) conditions</u>

(f_1) $$\lim_{m \to \infty} a_{mnij} = a_{.nij} \qquad \text{for all } n ,$$

$$\lim_{n \to \infty} a_{mnij} = a_{m.ij} \qquad \text{for all } m , \quad (i, j = 1, 2, \ldots) ,$$

(f_2) $$\lim_{m \to \infty} \sum_{i=1}^{\infty} a_{mnij} = L_{.n.j} \qquad \text{for all } n , \quad (j = 1, 2, \ldots) ,$$

$$\lim_{m \to \infty} \sum_{j=1}^{\infty} a_{mnij} = L_{.ni.} \qquad \text{for all } n , \quad (i = 1, 2, \ldots) ,$$

$$\lim_{n \to \infty} \sum_{i=1}^{\infty} a_{mnij} = L_{m..j} \qquad \text{for all } m , \quad (j = 1, 2, \ldots) ,$$

$$\lim_{n \to \infty} \sum_{j=1}^{\infty} a_{mnij} = L_{m.i.} \quad \text{for all } m, \quad (i = 1, 2, \ldots),$$

(f$_3$) $$\lim_{m \to \infty} \sum_{i,j=1}^{\infty} a_{mnij} = L_{.n..} \quad \text{for all } n,$$

$$\lim_{n \to \infty} \sum_{i,j=1}^{\infty} a_{mnij} = L_{m...} \quad \text{for all } m,$$

(f$_4$) There exist numbers b_{mij} such that

$$\lim_{n \to \infty} \sum_{i=1}^{\infty} |a_{mnij} - b_{mij}| = 0 \quad \text{for all } m, \quad (j = 1, 2, \ldots),$$

$$\lim_{n \to \infty} \sum_{j=1}^{\infty} |a_{mnij} - b_{mij}| = 0 \quad \text{for all } m, \quad (i = 1, 2, \ldots),$$

and numbers b_{nij} such that

$$\lim_{m \to \infty} \sum_{i=1}^{\infty} |a_{mnij} - b_{nij}| = 0 \quad \text{for all } n, \quad (j = 1, 2, \ldots),$$

$$\lim_{m \to \infty} \sum_{j=1}^{\infty} |a_{mnij} - b_{nij}| = 0 \quad \text{for all } n, \quad (i = 1, 2, \ldots).$$

(f$_5$) There exist numbers b_{mij} and b_{nij} such that

$$\lim_{n \to \infty} \sum_{i,j=1}^{\infty} |a_{mnij} - b_{mij}| = 0 \quad \text{for all } m, \text{ and}$$

$$\lim_{m \to \infty} \sum_{i,j=1}^{\infty} |a_{mnij} - b_{nij}| = 0 \quad \text{for all } n.$$

$(r\ c) \cap (r_o)$ conditions

$(\bar{f}_1)$ (f_1) , with

$$a_{.nij} = 0 \qquad \text{for all } n ,$$

$$a_{m.ij} = 0 \qquad \text{for all } m , \qquad (i,\ j = 1,\ 2,\ \ldots) ,$$

$(\bar{f}_2)$ (f_2) , with

$$L_{.n.j} = 0 \ ; \ L_{.ni.} = 0 \qquad \text{for all } n , \quad (i,\ j = 1,\ 2,\ \ldots) ,$$

$$L_{m..j} = 0 \ ; \ L_{m.i.} = 0 \qquad \text{for all } m , \quad (i,\ j = 1,\ 2,\ \ldots) ,$$

$(\bar{f}_3)$ (f_3) , with

$$L_{.n..} = 0 \ \text{ for all } n \text{ and } L_{m...} = 0 \ \text{ for all } m ,$$

$(\bar{f}_4)$ (f_4) , with

$$b_{mij} = 0 \qquad \text{for all } m ,$$

$$b_{nij} = 0 \qquad \text{for all } n , \quad (i,\ j = 1,\ 2,\ \ldots) ;$$

$(\bar{f}_5)$ (f_5) , with

$$b_{mij} = 0 \qquad \text{for all } m ,$$

$$b_{nij} = 0 \qquad \text{for all } n , \quad (i,\ j = 1,\ 2,\ \ldots) .$$

(a c), (a c_o), (a u r c_o), (a r c_o) conditions [25, 1948]

Combination of conditions selected from (a) to (f) together with the appropriate ones from the following:

$$(g_1) \qquad \Big| \sum_{i,j=1}^{p,q} a_{mnij} \Big| < I(m, n) \qquad (p, q, m, n = 1, 2, \ldots) ,$$

$$(g_2) \qquad \sum_{j=1}^{\infty} a_{mnij} = L_{mni.} \qquad (i, m, n = 1, 2, \ldots) ,$$

$$\sum_{i=1}^{\infty} a_{mnij} = L_{mn.j} \qquad (j, m, n = 1, 2, \ldots) ,$$

$$(g_3) \qquad \sum_{i,j=1}^{\infty} a_{mnij} = L_{mn..} \qquad (m, n = 1, 2, \ldots) ,$$

(g_4) (g_1) , with $I(m, n) = I$, $m, n \geq M$, $M \in J^+$,

(g_5) (g_4) , with $M = 1$.

With the conditions on the matrix established in § 6 , answers to the first four questions of § 3 are almost completed. The answers are given in the form of theorems over 150 in number. Contributions were made by the authors mentioned previously as well as Nigam (1939) and Mears (1948). We state only some of the significant results. For detailed statements of the theorems, the reader is referred to [14, 1936], [28, 1940] and [25, 1948].

Theorem 2.6.1 (a) [14, 1936]

A necessary condition that method A transform bounded sequences into convergent sequences is

(d_5) There exist numbers b_{ij} such that

$$\lim_{m,n \to \infty} \sum_{i,j=1}^{\infty} |a_{mnij} - b_{ij}| = 0 \quad .$$

Theorem 2.6.1 (b)

The sufficient conditions that method A transform bounded sequences into convergent sequences are (d_5) and

(b_1) $$\sum_{i,j=1}^{\infty} |a_{mnij}| < P \qquad \text{for} \quad m, n > B \ .$$

Theorem 2.6.2 (a)

A necessary condition that method A transform bounded sequences into regularly convergent sequences is

(f_5) There exist numbers b_{mij} and b_{nij} such that

$$\lim_{n \to \infty} \sum_{i,j=1}^{\infty} |a_{mnij} - b_{mij}| = 0 \quad \text{for all} \quad m \ ,$$

$$\lim_{m \to \infty} \sum_{i,j=1}^{\infty} |a_{mnij} - b_{nij}| = 0 \quad \text{for all} \quad n \ .$$

<u>Theorem 2.6.2 (b)</u>

The sufficient conditions that method A transform bounded sequences into regularly convergent sequences are (f_5)

$$(d_1) \qquad \lim_{m,n \to \infty} a_{mnij} = a_{..ij} , \qquad (i, j = 1, 2, \ldots)$$

and

$$(c_1) \qquad \sum_{i,j=1}^{\infty} |a_{mnij}| < P \qquad (m, n = 1, 2, \ldots) .$$

<u>Theorem 2.6.3 (a)</u> [25, 1948]

A necessary condition that method A transform absolutely convergent sequences into convergent sequences is

$$(d_3) \qquad \lim_{m,n \to \infty} \sum_{i,j=1}^{\infty} a_{mnij} = L .$$

<u>Theorem 2.6.3 (b)</u>

The sufficient conditions that method A transform absolutely convergent sequences into convergent sequences are (d_1) , (d_3) and

$$(g_4) \qquad \left| \sum_{i,j=1}^{p,q} a_{mnij} \right| < I(m, n) < I ,$$

$$(p, q = 1, 2, \ldots) , \quad m, n \geq M , M \in J^+ .$$

(d_2) $$\lim_{m,n\to\infty} \sum_{i=1}^{\infty} a_{mnij} = L_{\ldots j} ,$$

$$\lim_{m,n\to\infty} \sum_{j=1}^{\infty} a_{mnij} = L_{\ldots i \ldots} , \quad (i, j = 1, 2, \ldots) .$$

<u>Theorem 2.6.4</u> [19, 1922]

The necessary and sufficient conditions that method A transform regularly convergent sequences into convergent sequences are

$(\bar{d}_1)$ $$\lim_{m,n\to\infty} a_{mnij} = 0 \quad (i, j = 1, 2, \ldots) ,$$

(b_2) $$a_{mnij} = 0 \quad \text{for all } m, n, i > M_j , \ (j = 1, 2, \ldots), \text{ and for all } m, n, j > M_i , \ (i = 1, 2, \ldots), \text{ for some } M_i , M_j \in J^+ ,$$

(c_1) $$\sum_{i,j=1}^{\infty} |a_{mnij}| < P \quad (m, n = 1, 2, \ldots)$$

and

(a_3) $$\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} a_{mnij} = 1 .$$

7. <u>Recent Results</u>

In this section we consider the answers to the fifth question on the summability of double sequences and series. We remark also that functional analysis is applied in the proofs of all the results given in this section.

As early as 1940, Hill was able to extend the results of Banach on perfect summability of simple sequences [2] to certain cases of double sequence summability and established the following counterpart of Banach's theorem.

<u>Theorem 2.7.1</u> [17, 1940]

> Let the method A be perfect, and let the method B transform (r c) into (c) regularly and transform into (c) every sequence that A transforms into (r c) . Then every sequence $\{s_{ij}\}$ transformed into (r c) by A is transformed into (c) by B with the same principal limit. Moreover, if B is completely regular and transforms into (r c) every sequence that A does, then the row and column limits assigned by B to $\{s_{ij}\}$ will be respectively equal to those assigned by A .

While Hill gives the relation between a perfect summability and regular summability method and perfect summability and completely regular summability method, he does not give the conditions for which the method may transform a regularly convergent sequence into a regular, completely regular or perfect sequence.

In 1955, Alexiewicz and Orlicz [6, 1955] establish the necessary and sufficient conditions for the method A to transform regularly convergent sequences into regularly convergent sequences and completely regular

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(2) Banach, Theorie des operations lineaires, p. 95.

convergent sequences. Ramanujan [29, 1958] establishes the necessary and sufficient conditions for the method A to transform perfectly convergent sequences into perfectly convergent sequences. Alexiewicz and Orlicz also improve the consistence theorem of Hill by requiring the method A to be completely permanent instead of being perfect. Furthermore, they establish results equivalent to those obtained by using the A-method by means of three-dimensional matrix transformations on regularly convergent sequences. Thus question (v) is completely answered.

Remark 2.7.1

In order to keep this section completely self-contained, we allow some repetition of the conditions from the previous section.

A. The A-method of four-dimensional matrices

For the A-method, we have the following conditions for summability.

Theorem 2.7.2

The method A transforms regularly every sequence regularly convergent to 0 if and only if the following conditions are satisfied:

$$(a_1) \quad \sup_{m,n = 1, 2, \ldots} \sum_{i,j=1}^{\infty} |a_{mnij}| < \infty \; ,$$

$$(d_1) \quad \lim_{m,n \to \infty} a_{mnij} = a_{..ij} \; ,$$

(f_1) $$\lim_{n \to \infty} a_{mnij} = a_{m.ij} \quad , \quad \text{and}$$

$$\lim_{m \to \infty} a_{mnij} = a_{.nij}$$

(d_2) $$\lim_{m,n \to \infty} \sum_{j=1}^{\infty} a_{mnij} = L_{..i.} \quad , \quad \text{and}$$

$$\lim_{m,n \to \infty} \sum_{i=1}^{\infty} a_{mnij} = L_{...j} \quad ,$$

(f_2) $$\lim_{m \to \infty} \sum_{i=1}^{\infty} a_{mnij} = L_{.n.j} \quad ,$$

$$\lim_{m \to \infty} \sum_{j=1}^{\infty} a_{mnij} = L_{.ni.} \quad ,$$

$$\lim_{n \to \infty} \sum_{i=1}^{\infty} a_{mnij} = L_{m..j} \quad ,$$

$$\lim_{n \to \infty} \sum_{j=1}^{\infty} a_{mnij} = L_{m.i.} \quad .$$

If these conditions are satisfied, then for every sequence $\{s_{ij}\}$ regularly convergent to 0 the following formulae hold:

$$\sigma_{m.} = \sum_{i,j=1}^{\infty} a_{m.ij} s_{ij} + \sum_{i=1}^{\infty} s_{i.} \left(L_{m.i.} - \sum_{j=1}^{\infty} a_{m.ij} \right)$$

$$+ \sum_{j=1}^{\infty} s_{.j} \left(L_{m..j} - \sum_{i=1}^{\infty} a_{m.ij} \right) \; ;$$

$$\sigma_{.n} = \sum_{i,j=1}^{\infty} a_{.nij}\, s_{ij} + \sum_{i=1}^{\infty} s_{i.} \quad (L_{.ni.} - \sum_{j=1}^{\infty} a_{.nij})$$

$$+ \sum_{j=1}^{\infty} s_{.j} \quad (L_{.n.j} - \sum_{i=1}^{\infty} a_{.nij}) \ ;$$

$$\sigma_{.j} = \sum_{i,j=1}^{\infty} a_{..ij} s_{ij} + \sum_{i=1}^{\infty} s_{i.} \quad (L_{..i.} - \sum_{j=1}^{\infty} a_{..ij})$$

$$+ \sum_{j=1}^{\infty} s_{.j} \quad (L_{...j} - \sum_{i=1}^{\infty} a_{..ij}) \ ;$$

moreover

$$\lim_{m \to \infty} a_{m.ij} = \lim_{n \to \infty} a_{.nij} = a_{..ij} \ ,$$

$$\lim_{m \to \infty} L_{m.i.} = \lim_{n \to \infty} L_{.ni.} = L_{..i.} \ ,$$

$$\lim_{m \to \infty} L_{m..j} = \lim_{n \to \infty} L_{.n.j} = L_{...j} \ .$$

<u>Theorem 2.7.3</u>

The method A transforms regularly every sequence regularly convergent to 0 into a sequence convergent to 0 if and only if the conditions (a_1) , (f_1) , (f_2) are satisfied and

$$(\bar{d}_1) \quad \lim_{m,n \to \infty} a_{mnij} = a_{..ij} = 0 \ ,$$

$$(\bar{d}_2) \quad \lim_{m,n\to\infty} \sum_{i=1}^{\infty} a_{mnij} = L_{...j} = 0 \quad \text{and}$$

$$\lim_{m,n\to\infty} \sum_{j=1}^{\infty} a_{mnij} = L_{..i.} = 0 \ .$$

Theorem 2.7.4

The method A fulfills the condition (r_o) of definition 2.2.6 if and only if the conditions (a_1) , $(\bar{d}_1)$, $(\bar{d}_2)$, (f_1) and (f_2) are satisfied and

$$(\bar{e}_1) \quad \lim_{n\to\infty} a_{mnij} = a_{m.ij} = 0 \ ,$$

$$(h_1) \quad L_{m..i} = L_{.mi.} = 0 \ ,$$

$$(h_2) \quad L_{m.i.} = L_{.m.i} = \delta_{mi} \ ,$$

where δ_{mi} denotes the delta of Kronecker.

Theorem 2.7.5

The method A transforms regularly every sequence regularly convergent if and only if conditions (a_1) , (d_1) , (d_2) , (f_1) , (f_2) and the following are satisfied;

$$(d_3) \quad \lim_{m,n\to\infty} \sum_{i,j=1}^{\infty} a_{mnij} = L \ ,$$

$$(e_3) \quad \lim_{m\to\infty} \sum_{i,j=1}^{\infty} a_{mnij} = L_{.n..} \ , \quad \text{and}$$

$$\lim_{n\to\infty} \sum_{i,j=1}^{\infty} a_{mnij} = L_{m...} \quad .$$

In this case

$$\lim_{m\to\infty} L_{m...} = \lim_{n\to\infty} L_{.n..} = L \quad ,$$

$$\sigma_{m.} = s(L_{m...} - \sum_{i=1}^{\infty} L_{m.i.} - \sum_{j=1}^{\infty} L_{m..j}) + \sum_{i,j=1}^{\infty} a_{m.ij}\, s_{ij}$$

$$+ \sum_{i=1}^{\infty} (L_{m.i.} - \sum_{j=1}^{\infty} a_{m.ij})\, s_{i.}$$

$$+ \sum_{j=1}^{\infty} (L_{m..j} - \sum_{i=1}^{\infty} a_{m.ij})\, s_{ij} \quad ,$$

and similar formulae hold for $\sigma_{.n}$ and σ .

<u>Theorem 2.7.6</u>

The method A is completely permanent if and only if conditions (a_1) , $(\bar{d}_1)$, $(\bar{d}_2)$, (d_3) , (e_3) , (f_2) , (h_1) , (h_2) and the following are satisfied:

$(\bar{f}_1)$ $\quad a_{.nij} = a_{m.ij} = 0 \quad ,$

(h_3) $\quad L = L_{m...} = L_{.n..} = 1 \quad .$

We now give the generalized result of Hill due to Alexiewicz and Orlicz [6, 1955].

Theorem 2.7.7

Let the method A be completely regular and let every regularly convergent sequence be B-summable to its limit. If every bounded sequence $\{s_{ij}\}$ regularly A-summable is B-summable, then

$$A - \lim_{i,j \to \infty} s_{ij} = B - \lim_{i,j \to \infty} s_{ij} .$$

Theorem 2.7.8 [29, 1958]

The method A transforms all sequences perfectly convergent to 0 into perfectly convergent sequences if and only if

(a_1) $\sup_{m,n = 1,2,\dots} \sum_{i,j=1}^{\infty} |a_{mnij}| < \infty$.

(i_1) For each fixed i and j, the double sequence $\{a_{mnij}\}$ is perfectly convergent.

Theorem 2.7.9

The method A is completely regular for sequences perfectly convergent to 0 if and only if it satisfies (a_1) and

(i_2) For fixed i and j, the double sequence $\{a_{mnij}\}$ is perfectly convergent to 0 .

Theorem 2.7.10

The method A is completely conservative for perfectly convergent sequences if and only if (a_1) and the following conditions are satisfied:

(i_3) For fixed i and j, the sequence $\{a_{mnij}\}$ is perfectly convergent with limit $L_{..ij}$.

(i_4) $L_{mn..} = \sum_{i,j=1}^{\infty} a_{mnij}$ is perfectly convergent with limit L.

Under these conditions the limit of the transformed sequence will be

$$s \left[L - \sum_{i,j=1}^{\infty} L_{..ij} \right] + \sum_{i,j=1}^{\infty} L_{..ij}\, s_{ij} ,$$

where $s = \lim_{i,j \to \infty} s_{ij}$.

Theorem 2.7.11

The method A is completely regular for perfectly convergent sequences if and only if (a_1), (i_2) and the following condition are satisfied:

(i_5) $L_{mn..} = \sum_{i,j=1}^{\infty} a_{mnij}$ is perfectly convergent with limit 1.

B. Transformations reducing the degree of multiplicity of the given sequence [6, 1955]

The double-to-single sequence transformation is defined by a three-dimensional matrix

$$M : (a_{ijk}) , \qquad i, j, k = 0, 1, 2, \ldots$$

with the transforms

$$M_i(s) = \sum_{j,k=0}^{\infty} a_{ijk} \; s_{jk} \; .$$

Such methods map double sequences upon single ones. M-summability is defined as usual and the generalized limit is written

$$M - \lim_{j,k \to \infty} s_{jk} \quad \text{or simply } \sigma \; .$$

Definition 2.7.1

The method M is said to fulfill the condition (p_o) if it transforms every sequence perfectly convergent to 0 into a convergent sequence if and only if the following conditions are satisfied:

(α_1) $\sup\limits_i \sum_{j,k=0}^{\infty} |a_{ijk}| < \infty$.

(α_2) There exists

$$\lim_{i \to \infty} a_{ijk} = a_{.jk} \; , \quad (j, k = 0, 1, 2, \ldots) \; .$$

Theorem 2.7.12

The method M satisfies the condition (p_o) if and only if the conditions (α_1) and (α_2) are satisfied and

$$(\alpha_3) \qquad \lim_{i \to \infty} a_{ijk} = a_{.jk} = 0 \quad .$$

Theorem 2.7.13

The method M transforms every sequence regularly convergent into a convergent sequence if and only if the conditions (α_1), (α_2) and the following are satisfied:

$$(\beta_1) \qquad \lim_{i \to \infty} \sum_{j=0}^{\infty} a_{ijk} = L_{..k} \quad ,$$

$$\lim_{i \to \infty} \sum_{k=0}^{\infty} a_{ijk} = L_{.j.} \quad ,$$

$$\lim_{i \to \infty} \sum_{j,k=0}^{\infty} a_{ijk} = L \quad .$$

Theorem 2.7.14

The method M transforms every regularly convergent sequence into a sequence convergent to the same limit if and only if the conditions (α_1), (α_2), (β_1) are satisfied and

$$(\beta_2) \qquad a_{.jk} = L_{..k} = L_{.j.} = 0 \quad ,$$

$$(\beta_3) \qquad L = 1 \quad .$$

If the method M satisfies the conditions (α_1) , (α_2) and (β_1) , then

$$\sigma = \sum_{j,k=0}^{\infty} a_{.jk} \; s_{jk} + \sum_{j=0}^{\infty} (L_{.j.} - \sum_{k=0}^{\infty} a_{.jk}) \, s_{j.}$$

$$+ \sum_{k=0}^{\infty} (L_{..k} - \sum_{j=0}^{\infty} a_{.jk}) \, s_{.k}$$

for every regularly convergent sequence.

<u>Definition 2.7.2</u>

The number

$$c(M) = L - \sum_{j,k=0}^{\infty} a_{.jk}$$

is called the characteristic of the method M.

Let $M' : (a'_{ijk})$ be another method satisfying the conditions (α_1) , (α_2) and (β_1) ; for this method denote the numbers defined by the conditions (α_2) and (β_1) by $a'_{.jk}$, $L'_{..k}$, $L'_{.j.}$, L' respectively. Then we have

<u>Theorem 2.7.15</u>

The methods M and M' are consistent for regularly convergent sequences if and only if

$$a_{.jk} = a'_{.jk} \; ; \quad L_{.j.} = L'_{.j.} \; ; \quad \text{and} \quad L = L' ,$$

for $j, k = 0, 1, 2, \ldots$, and in this case

$$c(M) = c(M') \quad .$$

Theorem 2.7.16

Let $c(M) \neq 0$ and let the methods P and Q be consistent for regularly convergent sequences. If every bounded P-summable sequence is Q-summable, then

$$P - \lim_{i,j \to \infty} s_{ij} = Q - \lim_{i,j \to \infty} s_{ij} \quad .$$

C. Restrictedly convergent double sequences

We conclude this chapter by stating the consistence theorem on summability methods for restrictedly convergent double sequences due to Alexiewicz and Orlicz [7, 1959]. The conditions on the matrix A, for the restrictedly convergent sequences being analogous to those for the regularly convergent sequences, are omitted. The reader is referred to [7, 1959] for further detail.

Notation

As before $(b) \cap (c)$ is used to denote the class of all bounded convergent sequences while $[(b) \cap (c)]$ is now introduced to denote the class of all bounded restrictedly convergent sequences. $(b) \cap (c_o)$ and $[(b) \cap (c_o)]$ will stand for the subspaces of $(b) \cap (c)$ and $[(b) \cap (c)]$, composed of the bounded null convergent and bounded restrictedly null convergent sequences respectively.

Let Z be any class of bounded convergent or bounded restrictedly convergent sequences.

Definition 2.7.3

The method A is called restrictedly conservative (abbreviated r-conservative) for Z if it transforms every sequences of Z into a bounded restrictedly convergent sequence.

Definition 2.7.4

The method A is said to be restrictedly permanent (abbreviated r-permanent) for Z if it is r-conservative for the class and the generalized limit σ is equal to the ordinary limit (or the limit in the restricted sense) for every $\{s_{ij}\} \in Z$.

Theorem 2.7.17

Let the methods A and B be r-permanent for $(b) \cap (c_o)$ and let every bounded sequence $\{s_{ij}\}$ restrictedly A-summable to 0 be restrictedly B-summable. Then

$$B - \left[\lim_{i,j \to \infty} \right] s_{ij} = 0 \quad .$$

Definition 2.7.5

The number

$$c(A) = L - \sum_{i,j=1}^{\infty} a_{..ij}$$

defined for methods r-conservative for $(b) \cap (c)$ is called the characteristic of A .

Definition 2.7.6

The methods A and B are called r-consistent for the class Z of sequences if each sequence of Z is restrictedly summable by both methods to the same value.

Theorem 2.7.18

Let the methods A and B be r-consistent for $(b) \cap (c)$ and let $c(A) \neq 0$. If each bounded restrictedly A-summable sequence $\{s_{ij}\}$ is restrictedly B-summable, then

$$A - \left[\lim_{i,j \to \infty}\right] s_{ij} = B - \left[\lim_{i,j \to \infty}\right] s_{ij} .$$

CHAPTER III

SUMMABILITY OF DOUBLE SEQUENCES AND SERIES (II)

(SPECIAL METHODS)

It is the purpose of the author to state, in this chapter, the various methods of summing double sequences and series and the relations between these methods whenever such relations exist. References corresponding to the methods are also given so that the reader may refer to them for further information.

1. Definitions

We state the important definitions of this chapter in their generalized forms.

Let G_{mn} be the transform of a given double series $\sum_{i,j=0}^{\infty} u_{ij}$ with partial sums

$$s_{\mu\nu} = \sum_{i,j=0}^{\mu,\nu} u_{ij}$$

by some method of summation (G).

Definition 3.1.1

The double series $\sum_{i,j=0}^{\infty} u_{ij}$ is said to be restrictedly (G) summable (or (G_λ)-summable) if the sequence $G_{mn} \to s$ as $m, n \to \infty$ under the condition $\lambda^{-1} \leq m/n \leq \lambda$, where $\lambda > 1$ arbitrary but fixed.

Definition 3.1.2

The double series $\sum_{i,j=0}^{\infty} u_{ij}$ is said to be absolutely (G)-summable, denoted by $|(G)|$-summable, if

$$\sum_{m,n=1}^{\infty} \left|G_{mn} - G_{m-1,n} - G_{m,n-1} + G_{m-1,n-1}\right| < \infty ,$$

and if

$$\sum_{m=1}^{\infty} \left|G_{mn} - G_{m-1,n}\right| < \infty , \text{ and } \sum_{n=1}^{\infty} \left|G_{mn} - G_{m,n-1}\right| < \infty ,$$

for each n and m respectively.

Definition 3.1.3

Let $p > 0$ and let A be a matrix (a_{mnij}) with elements defined when $1 \leq i \leq m < \infty$ and $1 \leq j \leq n < \infty$.

(i) A double series $\sum u_{ij}$ with partial sums $s_{mn} = \sum_{i,j=0}^{m,n} u_{ij}$ is said to be strongly (A)-summable to s if

$$\lim_{m,n \to \infty} \sum_{i,j=0}^{m,n} a_{mnij} |s_{ij} - s|^p = 0 .$$

(ii) A double series $\sum u_{ij}$ is said to be restrictedly strongly (A)-summable (or strongly (A_λ)-summable) to s if for each $\lambda > 1$,

$$\lim_{m,n \to \infty} \sum_{i,j=0}^{m,n} a_{mnij} |s_{ij} - s| = 0$$

subject to the restriction $\lambda^{-1} \leq m/n \leq \lambda$.

2. Special Summability Methods

3.2.1 Abel (A, λ_m, α_n) means [16, 1948], [19, 1943].

For a given double series $\sum_{i,j=0}^{\infty} u_{ij}$ and the sequences of numbers

$$0 \leq \lambda_o < \lambda_1 < \lambda_2 < \ldots \qquad \lambda_n \to \infty ,$$

$$0 \leq \alpha_o < \alpha_1 < \alpha_2 < \ldots \qquad \alpha_n \to \infty ,$$

if the series

$$\sum_{m,n=0}^{\infty} u_{mn} \, e^{-\lambda_m x} \, e^{-\alpha_n y}$$

is convergent for all $x, y > 0$, and if

$$\text{(a)} \qquad f(x, y) = \sum_{m,n=0}^{\infty} u_{mn} \, e^{-(\lambda_m x + \alpha_n y)} \to S$$

when $x, y \to \infty$, then we say that the series $\sum_{i,j=0}^{\infty} u_{ij}$ is summable (A, λ_m, α_n) , or (A, λ, α) , to the sum S ; and write

$$\sum_{i,j=0}^{\infty} u_{ij} = S(A, \lambda, \alpha) .$$

In particular, for $\lambda_m = m^\lambda$ and $\alpha_n = n^\alpha$, we have the (A, m^λ, n^α) means:

$$f(x, y) = \sum_{m,n=0}^{\infty} u_{mn} \, e^{-(m^\lambda x + n^\alpha y)} .$$

The (A, λ_m, α_n) method is regular. [59, 1951], [60, 1954].

As an application of Abel's method of summation, we give the following theorem due to Grazava [26, 1963].

Theorem 3.2.1

If the terms u_{mn} of a double series which is summable by Abel's method satisfy, for some fixed p and q $(0 < p < 1$, $0 < q < 1)$, the three conditions

(i) $m^p n^q u_{mn} \to 0$ for $m, n \to \infty$

(ii) $\sum_{i=m}^{\infty} |u_{in}| \to 0$ when $m \to \infty$ for all fixed n

(iii) $\sum_{j=n}^{\infty} |u_{mj}| \to 0$ when $n \to \infty$ for all fixed m ,

then the double series converges provided one of the two conditions (ii) and (iii) holds uniformly with respect to the fixed subscript n or m .

3.2.2 The generalized Barlaz $B_{\mu\nu}(x_\mu, y_\nu)$ means[1]

Let $\sum_{i,j=0}^{\infty} u_{ij}$ be a double series with its partial sums $s_{mn} = \sum_{i,j=0}^{m,n} u_{ij}$. The generalized Barlaz $B_{\mu\nu}(x_\mu, y_\nu)$ means are defined by the transformation

$$\text{(b)} \qquad B_{\mu\nu}(x_\mu, y_\nu) = e^{-(x_\mu + y_\nu)} \sum_{m,n=0}^{\mu,\nu} (s_{mn}\, x_\mu^m\, y_\nu^n / m!\, n!) \ .$$

If $\lim B_{\mu\nu}(x_\mu, y_\nu)$ exists as $x_\mu, y_\nu \to \infty$, $\mu, \nu \to \infty$, the sequence $\{s_{mn}\}$ is said to be $B_{\mu\nu}(x_\mu, y_\nu)$-summable.

Theorem 3.2.2

A necessary and sufficient condition that the method defined by (b) be regular is that

$$\lim_{\mu \to \infty} (x_\mu - \mu)/\sqrt{\mu} = -\infty$$

and

$$\lim_{\nu \to \infty} (y_\nu - \nu)/\sqrt{\nu} = -\infty \ .$$

The generalized Barlaz means are a variant of the Borel exponential means obtained by replacing the continuous parameters x

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(1) This method is the author's generalization of the result of J. Barlaz. On some triangular summation methods. Amer. J. of Math., 69(1947), 139-152.

and y by discrete variables x_μ and y_ν, respectively.

3.2.3 (a) Borel B_λ and B means

Let there be given a double series $\sum_{i,j=0}^{\infty} u_{ij}$ and its partial sums

$$U_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{n} u_{ij} .$$

Suppose that the double power series

$$U(x, y) = \sum_{i,j=0}^{\infty} U_{ij} \frac{x^i y^j}{i!j!}$$

converges for all values $x \geq 0$ and $y \geq 0$. Then the series $\sum_{i,j=0}^{\infty} u_{ij}$ is said to be B_λ-summable to sum S if

$$(c_1) \qquad \lim_{(x,y)_\lambda \to \infty} e^{-(x+y)} U(x, y) = S .$$

By the symbol $(x, y)_\lambda \to \infty$ we mean that $(x, y) \to \infty$ inside the sector defined by the inequalities

$$\lambda \leq y/x \leq \lambda , \quad 0 < \lambda < 1 .$$

The given series is said to be B-summable to sum S if

$$(c_2) \qquad \lim_{x,y \to \infty} e^{-(x+y)} U(x, y) = S ,$$

i.e. for each $\epsilon > 0$, there exists a number $T(\epsilon)$ such that

$$|e^{-(x+y)} U(x, y) - S| < \epsilon$$

for all $x > T$ and $y > T$.

For conditions on the sequence U_{mn} for which the series $\sum_{i,j=0}^{\infty}{}' u_{ij}$ is regularly B_λ-summable, the reader is referred to [12, 1947], [5, 1953] and [34, 1960].

3.2.3 (b) <u>Borel B'_λ and B' integral means</u>

Let the double power series

$$p(x, y) = \sum_{i,j=0}^{\infty}{}' u_{ij} \frac{x^i y^j}{i! j!}$$

converge for all values x and y (i.e. let $p(x, y)$ be an entire function). We write

$$\Phi(x, y) = \int_0^x \int_0^y e^{-(t+\tau)} p(t, \tau)\, dt\, d\tau .$$

Then the series $\sum_{i,j=0}^{\infty}{}' u_{ij}$ is said to be B'_λ-summable to the sum S if

$$(c_3) \qquad \lim_{(x,y)_\lambda \to \infty} \Phi(x, y) = S ,$$

and the series $\sum_{i,j=0}^{\infty}{}' u_{ij}$ is said to be B'-summable to the sum S if

$$(c_4) \qquad \lim_{x,y \to \infty} \Phi(x, y) = S \ .$$

Remark 3.2.1

The methods B and B' are limiting forms of the methods B_λ and B'_λ for $\lambda \to \infty$.

Remark 3.2.2

The reader is referred to Muraev [39, 1961] for the necessary and sufficient conditions for which an Euler (E, p, q)-summable series is summable by the Borel (B_λ)-exponential means.

3.2.4 de la Vallee-Poussin sums (V)

For a double series $\sum u_{ij}$, the de la Vallee-Poussin sums are defined by the series-to-sequence transform

$$(d_1) \qquad P_{mn} = \sum_{i,j=1}^{m,n} \frac{m!\ m!\ n!\ n!}{(m - i)!\ (m + i)!\ (n - j)!\ (n + j)!} u_{ij}$$

or for the sequence of partial sums $\{s_{\mu\nu}\}$ with $s_{\mu\nu} = \sum_{i,j=0}^{\mu,\nu} u_{ij}$,

by the sequence-to-sequence transform

$$(d_2) \qquad Q_{mn} = \sum_{\mu,\nu=1}^{m,n} \frac{m!\ m!\ n!\ n!\ (2\mu + 1)\ (2\nu + 1)}{(m + \mu + 1)!\ (m - \mu)!\ (n + \nu + 1)!\ (n - \nu)!} s_{\mu\nu} \ .$$

The methods defined by (d_1) and (d_2) are regular. In each case, the generalized sum is said to be the limit, if it exists, of the right hand side as $m, n \to \infty$.

Remark 3.2.3

In the case of single series, T. H. Gronwall and C. N. Moore have shown, independently, that any series summable by Cesàro's means of any order is also summable (V) to the same sum. It is the author's conjecture that the same will be true for double series.

3.2.5 Gronwall's (f_i, g_i)-method for double series[2]

Let the series $\sum_{i,j=0}^{\infty} u_{ij}$ be given. Define

$$w = \frac{4z}{(1-z)^2}$$

from which we see that the unit circle $|z| < 1$ is mapped simply on the w-plane slet along the real axis from 1 to $+\infty$, and consequently $|w| < 1$ is mapped simply on a region interior to $|z| < 1$, and $z = 0$ corresponds to $w = 0$.

We then define the functions $f_i(w)$ and $g_i(w)$ $(i = 1, 2)$ as follows:

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(2) This generalization is based on the results of T. H. Gronwall, Summation of series and conformal mapping. Annals of Math. (2) 33(1932), 101-117.

$f_i(w)$ is holomorphic for $|w| \leq 1$ except at $w = 1$, and

$$z = f_i(w)$$

maps $|w| < 1$ simply on a region D interior to $|z| < 1$ in such a manner that $z = 0$ corresponds to $w = 0$, and $z = 1$ to $w = 1$. The inverse function is holomorphic on the boundary of D except at $z = 1$, and at this point

$$1 - w = (1 - z)^{\lambda} (a + \dots) \quad , \quad \lambda \geq 1 \, , \; a > 0 \, ,$$

where the dots denote a power series in $1 - z$ without constant term (the coefficients may be complex). The $g_i(w)$ are defined by

$$g_i(w) = \sum_{n_i=0}^{\infty} b_{n_i} \, w^{n_i} \quad , \quad b_{n_i} \neq 0 \, , \; (i = 1, 2) \, ,$$

$$(n = 0, 1, 2, \dots) \, ,$$

and

$$g_i(w) = (1 - w)^{-\alpha} + \gamma_i(w) \quad , \quad \alpha > 0 \, , \; (i = 1, 2) \quad ,$$

where $\gamma_i(w)$ is holomorphic for $|w| \leq 1$, and finally

$$g_i(w) \neq 0 \qquad \text{for} \quad |w| < 1 \quad .$$

<u>Definition 3.2.1</u>

When the sequence $\{U_{mn}\}$ $(m, n = 0, 1, 2, \dots)$ is generated by the identity

$$\text{(e)} \qquad \sum_{i,j=0}^{\infty}{}' \, u_{ij} \, z^{i+j} = \frac{1}{g_i(w)} \sum_{m,n=0}^{\infty}{}' \, b_{mn} \, U_{mn} \quad w^{m+n} \quad ,$$

and

$$U_{mn} \to S \qquad \text{as} \quad m, n \to \infty ,$$

where

$$U_{mn} = \sum_{i,j=0}^{m,n} a_{mnij} u_{ij} \qquad (m,n = 0, 1, 2, \ldots) ,$$

then the double series $\sum_{i,j=0}^{\infty} u_{ij}$ is summable (f_i, g_i) to the sum S .

For a deeper investigation of the generalization of Gronwall's result, the reader is referred to [9, 1953] and [8, 1947].

3.2.6 Hausdorff means for double series

Let P be a matrix whose elements are

$$p_{mnk\ell} = (-1)^{k+\ell} \binom{m}{k} \binom{n}{\ell} , \qquad k \leq m , \ \ell \leq n$$

$$= 0 \qquad \text{otherwise,}$$

where m, n, k, ℓ are positive integers or zero. The so defined difference matrix P is its own inverse, i.e. $P = P^{-1}$. Let

$$U : (u_{mnk\ell})$$

be an arbitrary diagonal matrix, i.e. the elements

$$u_{mnk\ell} = 0$$

for $k \neq m$ or $\ell \neq n$ or both, so that the only non-zero elements are u_{mnmn} $(m, n = 0, 1, 2, \ldots)$. We write for simplicity

$$u_{mnmn} = u_{mn} .$$

Then the transformation matrix

$$A = P\,U\,P^{-1}$$

is called a Hausdorff matrix corresponding to the sequence $\{u_{mn}\}$.

We remark that Hausdorff matrices are commutative. A given double sequence $S = \{s_{mn}\}$ is said to be summable to σ in the Hausdorff sense corresponding to the sequence $\{u_{mn}\}$ if the sequence

$$H = \{h_{mn}\} \quad ,$$

where

$$H = AS$$

approaches σ as m, n become infinite. For the proof of the following theorem on regularity of Hausdorff means, the reader is referred to [2, 1933].

<u>Theorem 3.2.3</u>

The Hausdorff method of summability corresponding to the bounded sequence $\{u_{mn}\}$ is regular if and only if

$$(f_1) \qquad u_{mn} = \int_0^1 \int_0^1 u^m\, v^n\, d_u d_v g(u,v) \quad , \quad (m, n = 0, 1, 2, \ldots) \; ,$$

where $g(u, v)$ is of bounded variation in the sense of Hardy-Krause in $(0, 1) \times (0, 1)$ and

$$g(u,\ 0^+) = g(u,\ 0) = \lim_{v \to 0} g(u,\ v) = 0\ ,\quad 0 \leq u \leq 1\ ,$$

$$g(0^+,\ v) = g(0,\ v) = \lim_{u \to 0} g(u,\ v) = 0\ ,\quad 0 \leq v \leq 1\ ,$$

$$g(1,\ 1) - g(1,\ 0) - g(0,\ 1) + g(0,\ 0) = 1\ .$$

As a result of this theorem, the definition of regular Hausdorff means for double sequences may be put into a more applicable form.

<u>Theorem 3.2.4</u>(3) [28, 1933]

The Hausdorff transform $\{h_{mn}\}$ of a sequence $\{s_{mn}\}$ may be defined by

$$(f_2)\quad h_{mn} = \sum_{k,\ell=0}^{m,n} \binom{m}{k} \binom{n}{\ell} s_{k\ell} \int_0^1 \int_0^1 u^k (1-u)^{m-k}$$

$$v^\ell (1-v)^{n-\ell}\, d_u d_v g(u,v)\ .$$

<u>Proof</u>

Let $s_{k\ell} = x^k y^\ell$. Then

$$h_{mn} = \sum_{k,\ell=0}^{m,n} (-1)^{k+\ell} \binom{m}{k} \binom{n}{\ell} u_{k\ell} \sum_{r,s=0}^{k,\ell} (-1)^{r+s} \binom{k}{r} \binom{\ell}{s} x^r y^s$$

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(3) Ustina, F., Gibbs phenomenon and Lebesque constants for the Hausdorff means of double series. Univ. of Alberta thesis (1966).

$$= \sum_{k,\ell=0}^{m,n} (-1)^{k+\ell} \binom{m}{k}\binom{n}{\ell} u_{k\ell} (1 - x)^k (1 - y)^\ell$$

$$= \sum_{k,\ell=0}^{m,n} (-1)^{k+\ell} \binom{m}{k}\binom{n}{\ell} \int_0^1 \int_0^1 (u - ux)^k (v - vy)^\ell \, du \, dv \, g(u, v)$$

$$= \int_0^1 \int_0^1 \sum_{k,\ell=0}^{m,n} (-1)^{k+\ell} \binom{m}{k}\binom{n}{\ell} (u - ux)^k (v - vy)^\ell \, du \, dv \, g(u, v)$$

$$= \int_0^1 \int_0^1 (1 - u + ux)^m (1 - v + vy)^n \, du \, dv \, g(u, v)$$

$$= \int_0^1 \int_0^1 \sum_{k,\ell=0}^{m,n} \binom{m}{k}\binom{n}{\ell} (1 - u)^{m-k} u^k x^k (1 - v)^{n-\ell} v^\ell y^\ell \, du \, dv \, g(u, v)$$

$$= \sum_{k,\ell=0}^{m,n} \binom{m}{k}\binom{n}{\ell} x^k y^\ell \int_0^1 \int_0^1 u^k(1 - u)^{m-k} v^\ell(1 - v)^{n-\ell} \, du \, dv \, g(u, v) .$$

Hence

$$h_{mn} = \sum_{k,\ell=0}^{m,n} \binom{m}{k}\binom{n}{\ell} s_{k\ell} \int_0^1 \int_0^1 u_k(1 - u)^{m-k} v^\ell(1 - v)^{n-\ell} \, du \, dv \, g(u, v)$$

and theorem 3.2.4 is proved.

The reader is referred to the work of Bendukidze [4, 1952] for information on Hausdorff strong summability and restrictedly strong summability.

Remark 3.2.4

The regular Hausdorff method is a generalization of the Cesàro (C, α, β) means, the Euler (E, p, q) means and the Hölder (H, α, β) means.

The Cesàro (C, α, β) means are obtained by taking

$$u_{mn} = \binom{m+\alpha}{\alpha}^{-1} \binom{n+\beta}{\beta}^{-1}$$

for the elements of the transformation matrix (u_{mn}), or by taking

$$g(u, v) = [1 - (1 - u)^{\alpha}]\,[1 - (1 - v)^{\beta}]$$

for the generating function of the constants u_{mn}.

The Euler (E, p, q) means are obtained by letting the generating function

$$g(u, v) = \begin{cases} 1 & \text{for } p \leq u < 1 \text{ and } q \leq v < 1 , \\ 0 & \text{otherwise.} \end{cases}$$

Lastly, we have the Hölder (H, α, β) means by taking

$$g(u, v) = [\Gamma(\alpha)\,\Gamma(\beta)]^{-1} \left(\int_0^u (\log \tfrac{1}{x})^{\alpha-1}\, dx \right) \left(\int_0^v (\log \tfrac{1}{y})^{\beta-1}\, dy \right) .$$

For the sake of completeness, we shall describe each of the three special cases separately.

3.2.7 <u>Cesàro (C, α, β) means for double series</u> [55, 1955], [11, 1947].

We give here the Cesàro means of distinct order, i.e. $\alpha \neq \beta$. Let a double sequence $\{s_{mn}\}$ be formed from the double series $\sum_{i,j=0}^{\infty} u_{ij}$ with

$$s_{mn} = \sum_{i,j=0}^{m,n} u_{ij} \quad .$$

Define

$$\sigma_{m,n}^{(\alpha,\beta)} = \sum_{i,j=0}^{m,n} A_{m-i}^{\alpha-1} \; A_{n-j}^{\beta-1} \; s_{ij}$$

where α, β are real numbers greater than -1, and

$$A_n^{\alpha} = \binom{n+\alpha}{n} \quad .$$

Then the Cesàro matrix $C_{m,n}^{\alpha,\beta}$ may be defined by

$$\text{(g)} \qquad C_{m,n}^{\alpha,\beta} = \sigma_{m,n}^{(\alpha,\beta)} / A_m^{\alpha} A_n^{\beta} = y_{mn} \quad .$$

<u>Definition 3.2.2</u>

The series $\sum_{i,j=0}^{\infty} u_{ij}$ is said to be (C, α, β)-summable if the sequence $\{y_{mn}\}$ of (g) is convergent to some value s.

The Cesàro (C, α, β) means are regular for $\alpha, \beta \geq 0$.

The special case of the method (C, α, β) in which $\alpha = \beta = 1$ is very widely used. We give, as an example, the following theorem. For an example of the application of the $(C, 1, 1)$ method to double orthogonal series, the reader is referred to Fedulov [25, 1955].

Theorem 3.2.5

Let $\sum u_{ij}$ be a double series convergent to s and having partial sums $s_{mn} = \sum_{i,j=0}^{m,n} u_{ij}$ such that

(i) for each n, $s_{mn}/m \to 0$ as $m \to \infty$, and

(ii) for each m, $s_{mn}/n \to 0$ as $n \to \infty$.

Then the series is $(C_\lambda, 1, 1)$-summable to s.

Finally, using the corresponding meaning of absolute summability as defined by definition 3.1.2 for the (C, α, β) method, we obtain

Theorem 3.2.6 [57, 1950]

If $\alpha, \beta > -1$ and $h, k > 0$, then each series $\sum u_{ij}$ summable $|(C, \alpha, \beta)|$ is also summable $|(C, \alpha + h, \beta + k)|$.

We give here a brief account of the relation between Cesàro's method and the method of Abel.

Definition 3.2.3

A double series $\sum u_{ij}$ with partial sums s_{mn} is said to be $(C, 1, 1)$-summable to L if

$$\sigma_{mn} = (m + 1)^{-1} (n + 1)^{-1} \sum_{i,j=0}^{m,n} s_{ij} \to L \quad \text{as} \quad m, n \to \infty \ .$$

Definition 3.2.4

A double series $\sum u_{ij}$ with partial sums s_{mn} is said to be summable to L by the Euler-Abel power series method(4) if

$$f(x, y) = \sum_{i,j=0}^{\infty} s_{ij} x^i y^j \to L \quad \text{as} \quad x, y \to 1^- \ .$$

Ogieveckiĭ, [41, 1947], established the following result.

Theorem 3.2.7

If $\sum u_{ij}$ is summable $(C, 1, 1)$ to L and if the partial sums are such that

$$\lim_{m \to \infty} |s_{mn}|/m < \infty \ ,$$

$$\lim_{n \to \infty} |s_{mn}|/n < \infty \ ,$$

---

(4) Hille, E., Analytic function theory, Ginn and Company, (1959), p. 120.

for each n and m, respectively, then, for each $\lambda > 1$, $f(x, y) \to L$ as $x, y \to 1$ over $0 < x < 1$ and $0 < y < 1$ subject to the restriction $\lambda^{-1} \leq (1 - x)/(1 - y) \leq \lambda$.

For a more extensive treatment of this subject, the reader is referred to Celidze [10, 1947] and Cesari [1942].

For the relation between the Cesàro (C, α, β) means and Abel (A, λ_m, α_n) means, Žak and Timan [59, 1951] showed that a series absolutely (C, α, β)-summable, $(\alpha > -1, \beta > -1)$, is also absolutely summable by the Abel means. And later on, Ogieveckiĭ [42, 1953], [43, 1954] and [44, 1958] proved that a (C, α, β)-bounded and restrictedly (C, α, β)-summable series is also restrictedly (A, λ_m, α_n)-summable.

3.2.8 <u>The Euler (E, p, q) method</u>

Suppose that the series $\sum u_{mn} x^{m+1} y^{n+1}$ converges to $f(x, y)$ for small x and y, that $p, q > 0$ and that

$$x = \frac{v}{1 - pv} \quad , \quad v = \frac{x}{1 + px} \quad ;$$

$$y = \frac{w}{1 - qw} \quad , \quad w = \frac{y}{1 + qy} \quad ;$$

so that $v = (1 + p)^{-1}$ and $w = (1 + q)^{-1}$ when $x = 1$ and $y = 1$ respectively. Then for small x, v, y and w,

$$(h)\quad f(x, y) = \sum_{m,n=0}^{\infty} u_{mn} \left(\frac{v}{1 - pv}\right)^{m+1} \left(\frac{w}{1 - qw}\right)^{n+1}$$

$$= \sum_{m,n=0}^{\infty} u_{mn} \sum_{k=m,\ell=n}^{\infty} \binom{k}{m} p^{k-m}\ v^{k+1} \binom{\ell}{n} q^{\ell-n}\ w^{\ell+1}$$

$$= \sum_{k,\ell=0}^{\infty} v^{k+1}\ w^{\ell+1} \sum_{m,n=0}^{k,\ell} \binom{k}{m}\binom{\ell}{n} p^{k-m}\ q^{\ell-n}\ u_{mn}$$

$$= \sum_{k,\ell=0}^{\infty} u_{k\ell}^{(pq)} \left\{(p + 1)v\right\}^{k+1} \left\{(q + 1)w\right\}^{\ell+1} ,$$

where

$$u_{k\ell}^{(pq)} = (p + 1)^{-k-1}\ (q + 1)^{-\ell-1} \sum_{m,n=0}^{k,\ell} \binom{k}{m}\binom{\ell}{n} p^{k-m}\ q^{\ell-n}\ u_{mn} .$$

If $\sum_{k,\ell=0}^{\infty} u_{k\ell}^{(pq)} = L$, then we say that $\sum_{m,n=0}^{\infty} u_{mn}$ is summable (E, p, q) to L . The Euler (E, p, q) method is regular.

The reader is referred to [39, 1961] and [56, 1953] for further information concerning Euler's method of summation.

3.2.9 Hölder (H, α, β) means

Let $\{s_{mn}\}$ be a sequence of partial sums corresponding to the series $\sum_{i,j=0}^{\infty} u_{ij}$. We define the Hölder means $H^{i,k}$ of order (i, k) through the recurrence formula ([28, 1933]):

$$H^{0,0}_{m,n} = s_{mn} \quad ,$$

$$H^{0,k}_{m,n} = \frac{H^{0,k-1}_{m,0} + H^{0,k-1}_{m,1} + \dots + H^{0,k-1}_{m,n}}{n+1} \quad ,$$

$$H^{i,k}_{m,n} = \frac{H^{i-1,k}_{0,n} + H^{i-1,k}_{1,n} + \dots + H^{i-1,k}_{m,n}}{m+1} \quad .$$

The arithmetic means $H^{0,1}$ correspond to the relation

$$\eta_{mn} = \frac{1}{n+1}\, \xi_{mn} \quad ,$$

and these yield for the Hölder means $H^{0,k}$,

$$\eta_{mn} = \frac{1}{(n+1)^k}\, \xi_{mn} \quad ,$$

and in general

$$y_{mn} = H^{i,k}\,(s_{mn})$$

is synonymous with

$$\eta_{mn} = \frac{1}{(m+1)^i\,(n+1)^k}\, \xi_{mn} \quad .$$

This allows a definition of the Hölder means of every arbitrary order (α, β) . For $\alpha > 0$, we have the following integral representation:

$$\text{(i)} \qquad \frac{1}{(m+1)^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^1 u^m \,(\log \frac{1}{u})^{\alpha-1}\, du \quad ,$$

from which follows the regularity of the $H^{\alpha,0}$ means.

The Hölder means satisfy the relation

$$H^{\alpha,0} \cdot H^{0,\beta} = H^{\alpha,\beta} \quad ,$$

and so for $\alpha, \beta > 0$, $H^{\alpha,\beta}$ is regular. Since

$$H^{\alpha,\beta} \cdot H^{\delta,\gamma} = H^{\alpha+\delta,\beta+\gamma} \quad ,$$

it follows that the method is stronger with increasing α or β .

Lösch [31, 1942] establishes a formal relation between the three special cases of the regular Hausdorff summability method - Cesàro, Hölder and Euler-Knopp- for restrictedly summable sequences:

Let $\{A, B\}$ denote the double sequence transformation

$$\sigma_{mn} = \sum_{i,j=0}^{m,n} a_{mi}\, b_{nj}\, s_{ij} \quad ,$$

from s_{ij} to σ_{mn} , determined by the matrices A and B of real or complex constants a_{mi} and b_{nj} . We have the following theorems.

<u>Theorem 3.2.8</u>

If A_α is one of the matrices C_α, H_α, E_α of Cesàro, Hölder and Euler, respectively, and if r and t are non-negative integers, then a restrictedly convergent sequence $\{s_{ij}\}$ is summable $\{A_r, A_t\}$ if and only if it is restrictedly bounded $\{A_r, A_t\}$.

Theorem 3.2.9

If the sequence $\{s_{ij}\}$ is restrictedly convergent to L and restrictedly bounded $\{A_r, A_t\}$, then it is is restrictedly summable $\{A_r, A_t\}$ to L.

Theorem 3.2.10

If r and t are non-negative integers, a sequence $\{s_{ij}\}$ restrictedly summable $\{C_r, C_t\}$ is restrictedly summable $\{H_r, H_t\}$ if and only if it is restrictedly bounded $\{H_r, H_t\}$; and if $\{s_{ij}\}$ is restrictedly summable $\{C_r, C_t\}$ and $\{H_r, H_t\}$ to L_1 and L_2 respectively, then $L_1 = L_2$.

3.2.10 Quasi-Hausdorff means for double series

These means are defined by the transform

$$(j)\quad h^*_{mn}(x, y) = \sum_{\mu=m,\, \nu=m}^{\infty} \binom{\mu}{m}\binom{\nu}{n} s_{\mu\nu} \int_0^1\!\!\int_0^1 x^{m+1} (1 - x)^{\mu-m}$$

$$y^{n+1} (1 - y)^{\nu-n}\, d_x d_y\, \alpha(x, y) .$$

The transform is regular if and only if $\alpha(x, y)$ is of bounded variation in the sense of Hardy-Krause in $(0, 1) \times (0, 1)$ and

$$\alpha(x, 0^+) = \alpha(x, 0) = \lim_{y \to 0} \alpha(x, y) = 0 \quad 0 \leq x \leq 1 ,$$

$$\alpha(0^+, y) = \alpha(0, y) = \lim_{x \to 0} \alpha(x, y) = 0 \quad 0 \leq y \leq 1 ,$$

$$\alpha(1, 1) - \alpha(1, 0) - \alpha(0, 1) + \alpha(0, 0) = 1 .$$

3.2.11 Circle (γ, r, ℓ) means for double series(5)

Let $\{s_{mn}\}$ be the sequence of partial sums of the double series $\sum_{i,j=0}^{\infty} u_{ij}$. The circle (γ, r, ℓ) means of a given sequence are defined by the transformation

$$(k_1) \qquad C_{mn} = \sum_{\mu=m, \nu=n}^{\infty} \binom{\mu}{m} r^{m+1} (1 - r)^{\mu-m} \binom{\nu}{n} \ell^{n+1} (1 - \ell)^{\nu-n} s_{\mu\nu}$$

or

$$(k_2) \qquad C_{mn} = \sum_{\mu=m, \nu=n}^{\infty} \binom{\mu}{m}\binom{\nu}{n} r^{m+1} (1 - r)^{\mu-m} \ell^{n+1} (1 - \ell)^{\nu-n} s_{\mu\nu} .$$

The transform (k_2) is regular if and only if $0 < r \leq 1$ and $0 < \ell \leq 1$.

Remark 3.2.5

Whereas the Euler (E, p, q) means are obtained from the regular Hausdorff method by letting $g(u, v)$ of (f_2) take on the value defined by $g(u, v) = 1$ for $p \leq u < 1$ and $q \leq v < 1$ and

---

(5) This method is a generalization of the circle (γ, r) means given in the thesis of Ustina, F., Gibbs phenomenon and Lebesque constants.

$g(u, v) = 0$ otherwise; the circle (γ, r, ℓ) means result from substituting the value $g(u, v)$ defined above into the Quasi-Hausdorff transform (j) to replace $\alpha(x, y)$.

3.2.12 The Harmonic means for double series[6]

For the sequence $\{s_{mn}\}$ of partial sums of the series $\sum_{i,j=0}^{\infty} u_{ij}$, the Harmonic means for double series are defined by the transform

$$(\ell) \qquad H_{mn} = \{\log (mn)\}^{-1} \sum_{\mu,\nu=1}^{m,n} s_{m-\mu,n-\nu}/\mu\nu .$$

The method defined by (ℓ) is regular.

3.2.13 The Lambert means for double series (L)

The double series $\sum_{i,j=0}^{\infty} u_{ij}$ is said to be summable (L) to S, or that

$$s_{mn} = \sum_{i,j=0}^{m,n} u_{ij} \rightarrow S(L)$$

if

---

(6) This method is a generalization of the Harmonic means given in the thesis of Ustina, F., Gibbs phenomenon and Lebesque constants.

(m) $$F(x, y) = \sum u_{mn} \frac{(mx)(ny)\, e^{-(mx+ny)}}{1 - e^{-(mx+ny)}} \to S$$

when $x \to +0$ and $y \to +0$.

3.2.14 Lebesque's method

A double series $\sum u_{mn}$ is said to be summable to S by the Lebesque method if

(n) $$\lim_{x,y \to 0} F(x, y) = S$$

where

$$f(x, y) = \sum_{m,n=1}^{\infty} u_{mn} \left(\frac{\sin mx}{mx}\right)\left(\frac{\sin ny}{ny}\right) .$$

Tevzadze [49, 1953] shows that if

$$A_m = m \sum_{k=1}^{\infty} |u_{nk}| , \quad B_n = n \sum_{k=1}^{\infty} |u_{km}|$$

are convergent and if

$$\lim_{m \to \infty} A_m = 0 , \quad \lim_{n \to \infty} B_n = 0 ,$$

then $\sum_{m,n=1} u_{mn}$ is summable to S by the Lebesque method if and only if $\sum u_{mn}$ converges to S .

<u>Remark 3.2.6</u>

Lebesque's method is a special case of the Riemann (R, k, k) method with $k = 1$.

3.2.15 <u>Norlund $(N ; c)$ means for double series</u> [20, 1950] and [38, 1954]

Given a doubly infinite set of complex constants c_{ij} $(i, j = 0, 1, 2, \dots)$ where $c_{oo} \neq 0$. We set

$$C_{mn} = \sum_{i,j=0}^{m,n}{}' c_{ij} .$$

For any double series $\sum_{i,j=0}^{\infty}{}' u_{ij}$, where $s_{mn} = \sum_{i,j=0}^{m,n}{}' u_{ij}$, we form

$$(0) \qquad N_{mn} = \frac{S_{mn}}{C_{mn}} = \frac{\sum_{i,j=0}^{m,n} c_{m-i,n-j}\, s_{ij}}{C_{mn}} .$$

If N_{mn} tends to a limit L as m and n become infinite, we say that the series $\sum_{i,j=0}^{\infty}{}' u_{ij}$ is summable $(N ; c)$ to the value L. The necessary and sufficient condition for the regularity of the $(N ; c)$ method is that

$$\lim_{m,n \to \infty} \frac{c_{mn}}{C_{mn}} = 0 .$$

In the case when $c_{ij} = 1$, $i, j \geq 0$, N_{mn} becomes σ_{mn}, the Cesàro transformation. For a formal relation between the Norlund and Cesàro

methods of summation, the reader is referred to Chadaya [21, 1951].

3.2.16 Riemann (R, k, k) means for double series

A double series $\sum u_{mn}$ is summable to L by the Riemann method $(R, 2, 2)$ if

$$\lim_{x,y \to 0} f(x, y) = L ,$$

where

$$f(x, y) = \sum_{m,n=0}^{\infty} \left(\frac{\sin mx}{mx}\right)^2 \left(\frac{\sin ny}{ny}\right)^2 u_{mn} .$$

The $(R, 2, 2)$ method is regular. More generally, summability (R, k, k), where k is a positive integer, is defined by

$$\text{(p)} \qquad f(x, y) = \sum_{m,n=0}^{\infty} \left(\frac{\sin mx}{mx}\right)^k \left(\frac{\sin ny}{ny}\right)^k u_{mn}$$

and the double series is said to be summable (R, k, k) if in (p), we have

$$\lim_{x,y \to 0} f(x, y) = L .$$

The method defined by (p) is regular for $k > 1$.

For the conditions for which a given series is restrictedly (R, k, k)-summable, $k \geq 2$, the reader is referred to [58, 1952] and [24, 1962].

3.2.17 The (R_2) means

This method is closely related to the Riemann (R, k, k) means but not equivalent to it. The (R_2) means is defined by

$$(q) \qquad f(x, y) = \frac{2}{\pi} \sum_{m,n=0}^{\infty}{}' \left(\frac{\sin^2 mx}{m^2 x}\right)\left(\frac{\sin^2 ny}{n^2 y}\right) s_{mn} ,$$

where, as usual, $s_{mn} = \sum_{i,j=0}^{m,n}{}' u_{ij}$, and the coefficients of s_{oo} in the given sum is defined to be xy . The (R_2) means are also regular.

3.2.18 Riesz $(R : \lambda, p ; \mu, r)$ means for double series [33, 1928]

For a given double series $\sum_{k,\ell=0}^{\infty}{}' u_{k\ell}$, let $\{\lambda_n\}$ and $\{\mu_n\}$ be two sequences of real numbers, increasing and becoming infinite, with $\lambda_1 \geq 0$, $\mu_1 \geq 0$; let p and r be any real numbers, we define

$$C_{\lambda\mu}^{pr}(s, t) = \sum_{\lambda_k < s}{}' \sum_{\mu_\ell < t}{}' (s - \lambda_k)^p (t - \mu_\ell)^r u_{k\ell} .$$

Write

$$C_{\lambda\mu}(\sigma, \tau) = C_{\lambda\mu}^{oo}(\sigma, \tau) = \sum_{k,\ell=1}^{m,n}{}' u_{k\ell} \quad \text{for} \begin{cases} \lambda_m < \sigma \leq \lambda_{m+1} \\ \mu_n < \tau \leq \mu_{n+1} \end{cases} .$$

Then for $p > 0$, $r > 0$, we have

$$C_{\lambda\mu}^{pr}(s, t) = pr \int_0^s \int_0^t C_{\lambda\mu}(\sigma, \tau)\,(s - \sigma)^{p-1}(t - \tau)^{r-1}\, d\tau\, d\sigma \quad .$$

For $p = 0$, $r > 0$,

$$C_{\lambda\,\mu}^{pr}(s, t) = r \int_0^t C_{\lambda\mu}(s, \tau)\,(t - \tau)^{r-1}\, d\tau \quad ,$$

and for $r = 0$, $p > 0$,

$$C_{\lambda\mu}^{pr}(s, t) = p \int_0^s C_{\lambda\mu}(\sigma, t)\,(s - \sigma)^{p-1}\, d\sigma \quad .$$

The series $\sum_{k,\ell=1}^{\infty} u_{k\ell}$ is said to be summable $(R : \lambda, p ; \mu, r)$ to sum L if

$$(r) \qquad \lim_{s,t \to \infty} s^{-p}\, t^{-r}\, C_{\lambda\mu}^{pr}(s, t) = L \quad .$$

The Riesz $(R : \lambda, p ; \mu, r)$ means are regular for $p \geq 0$, $r \geq 0$.

Suppose the Cesàro transform of order (p, r) for the sequence $\{s_{ij}\}$ with $s_{ij} = \sum_{s,t=0}^{i,j} u_{st}$ is defined to be

$$\sigma_{m,n}^{(p,r)} = \sum_{i,j=0}^{m,n} \binom{m + p - i}{p} \binom{n - r - j}{r} s_{ij} \quad .$$

Let us put

$$A_{m,n}^{p,r} = \binom{m + p}{m}\binom{n + r}{n} .$$

By formula (g) of §3.2.7, we say that the series $\sum_{s,t=0}^{\infty}{}' u_{st}$

is summable (C, p, r) to the limit

$$(g') \qquad \lim_{m,n \to \infty} \sigma_{m,n}^{(p,r)} / A_{mn}^{pr} = L$$

if this limit exists. Merriman proved the following result.

<u>Theorem 3.2.11</u> [35, 1927]

The limit (g') and the limit (r) exist at the same time and are equal provided that the rows and columns of $\sum_{s,t=0}^{\infty}{}' u_{st}$ are summable as single series in either Cesàro or the Rieszian manners, indices r and p .

We now state the results of Riesz summability of the Dirichlet product and Cauchy product series of Riesz summable series. Since the Cauchy product is a special case of the Dirichlet product, we need only to concern ourselves with the results for the Dirichlet product, the same results being valid also for the Cauchy product.

For

$$A_{\lambda'\mu'}^{p'r'}(s, t) = \sum_{\lambda_m' < s} \sum_{\mu_n' < t} (s - \lambda_m')^{p'} (t - \mu_n')^{r'} a_{mn} ,$$

$$B_{\lambda''\mu''}^{p''r''}(s, t) = \sum_{\lambda_k'' < s} \sum_{\mu_\ell'' < k} (s - \lambda_k'')^{p''} (t - \mu_\ell'')^{r''} b_{k\ell} ,$$

and

$$C^{pr}_{\lambda\mu}(s, t) = \sum_{\lambda_i < s} \sum_{\mu_j < t} (s - \lambda_i)^p (t - \mu_j)^r c_{ij} ,$$

we have

<u>Theorem 3.2.12</u> [35, 1927]

If $\sum a_{mn}$ is summable $(R : \lambda', p' ; \mu', r')$ to A, and if $\sum b_{mn}$ is summable $(R : \lambda'', p'' ; \mu'', r'')$ to B, where p', r', p'' and r'' are greater than 1, and if $\sigma^{p'} \tau^{r'} A^{p'r'}_{\lambda'\mu'}(s, t)$ and $\sigma^{p''} \tau^{r''} B^{p''r''}_{\lambda''\mu''}(s, t)$ are bounded, then the Dirichlet product series, $\sum c_{mn}$, is summable $(R : \lambda, p ; \mu, r)$ to sum C, where $p = p' + p'' + 1$, $r = r' + r'' + 1$, and $AB = C$, and $C^{pr}_{\lambda\mu}(s, t)$ is bounded.

3.2.19 <u>The (S^*, α) and $S_{1-r,1-\ell}$ means for double series</u>

The (S^*, α) means, first introduced by Ramanujan(7), generalized for double series, are a form of the quasi-Hausdorff transformations for double series. This method is defined by the transform

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(7) Ramanujan, M.S., On Hausdorff and Quasi-Hausdorff methods of summability. Quart. J. Math. 8(1957), 197-213.

$$(s_1) \qquad S^*_{mn}(x, y) = \sum_{\mu,\nu=0}^{\infty} \binom{m+\mu}{\mu}\binom{n+\nu}{\nu} s_{\mu\nu}(x, y) \int_0^1\int_0^1 x^{m+1}$$

$$(1 - x)^{\mu}\, y^{n+1}\, (1 - y)^{\nu}\, dx\, dy\, \alpha(x, y) ,$$

where the weight function $\alpha(x, y)$ is of bounded variation in the sense of Hardy-Krause in $(0, 1) \times (0, 1)$. The transformation defined by (s_1) is regular if and only if the following two conditions are satisfied.

$$(i) \begin{cases} \alpha(x, 0^+) = \alpha(x, 0) = \lim\limits_{y \to 0} (x, y) = 0 \\ \alpha(0^+, y) = \alpha(0, y) = \lim\limits_{x \to 0} (x, y) = 0 \\ \alpha(1, 1) - \alpha(1, 0) - \alpha(0, 1) + \alpha(0, 0) = 1 \end{cases}$$

and

$$(ii) \begin{cases} \alpha(1, 1) = \alpha(1^-, 1^-) \\ \alpha(1, 0) = \alpha(1^-, 0) \\ \alpha(0, 1) = \alpha(0, 1^-) \quad . \end{cases}$$

The (S^*, α) means reduce to $S_{1-r,1-\ell}$ means defined by the transform

$$(s_2) \quad \sigma_{mn}(x, y) = r^{m+1}\, \ell^{n+1} \sum_{\mu,\nu=0}^{\infty} s_{\mu\nu}(x, y)\, (1 - \nu)^{\mu}(1 - \ell)^{\nu} \binom{m+\mu}{\mu}\binom{n+\nu}{\nu},$$

when $\alpha(x, y) = 1$ for $r \leq x < 1$, $\ell \leq y < 1$ and $\alpha(x, y) = 0$ otherwise. The transform (s_2) is regular if and only if $0 < r < 1$ and $0 < \ell < 1$.

CHAPTER IV

TOPICS IN DOUBLE SEQUENCES AND SERIES

We shall conclude our survey of the subject "Double Sequences and Series" with results on the following five topics in double series:

1. Tauberian theorems for double series
2. Convergence factors
3. Summability factors
4. Multiplication of double series
5. Results related to double sequences and series.

Due to the fact that many of the papers on these topics are not available to the author and that a detail study of any of these topics will make the thesis too lengthy, we give only the references following a brief remark on each topic.

1. Tauberian Theorems for Double Series

We have, in Chapters II and III, considered systematically the various types of the so-called "Abelian" theorems, i.e. theorems asserting the regularity of a method of summation. It is our aim, in this section, to concern ourselves with what Hardy called the "corrected forms of the false converses of Abelian theorems"[1]. In general, these theorems assert that if a series is summable (P) , and satisfies some

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(1) Hardy, G.H., Divergent series, Oxford Press, (1949), p. 149.

further conditions K_P (which will vary with a method P, but will in any case imply a certain slowness of possible divergence), then it is convergent. Such theorems are called 'Tauberian' after A. Tauber who first proved one of the simplest of them; and the supplementary condition is called a 'Tauberian condition'(2).

We give here a typical example of a Tauberian theorem in double series which is established by K. Knopp [9, 1939].

<u>Theorem 4.1.1</u>

Let

$$s_{mn} = \sum_{i,j=1}^{m,n} u_{ij} \quad ; \quad \sigma_{mn} = \frac{1}{mn} \sum_{i,j=1}^{m,n} s_{ij}$$

($m, n = 1, 2, \ldots$), denote the sequence of partial sums and the $(C_{1,1})$-transform of a real double series $\sum u_{mn}$. If $\sigma_{mn} \to s$ and $(m^2 + n^2)\, u_{mn} < K$, then $s_{mn} \to s$.

In 1940, H. D. Kloosterman [8, 1940] proved Tauberian theorems in double series for Cesàro-summability of any order. In that same year, R. P. Agnew [1, 1940] asserted that the conditions of the above theorem cannot be weaker. V. H. Durañonay [6, 1940] establishes results for bounded convergence from bounded summability and a Tauberian condition

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(2) Hardy, G.H., Divergent series, Oxford Press, (1949), p. 121 and p. 149.

parallel to the results of Knopp who deduces convergence from summability and a Tauberian condition.

Tauberian theorems in some particular double series are given by H. Delange [4, 1947] and [5, 1948]. The reader is referred to the following references for Tauberian theorems for the various methods of summation:

A. Euler methods

Berekašvili, V.A., On Euler methods of summation of double series. Soobšč. Akad. Nauk Gruzin. SSR 16 (1955), 337-342 (Russian).

B. Abel and Cesàro methods

Ogieveckiĭ, I.E., Some Tauberian theorems for double series. Dokl. Akad. Nauk SSSR (N.S.) 110(1956), 330-333 (Russian).

C. Cesàro methods

Čelidze, V.G., Tauberian theorems for multiple series.(Russian. Georgian summary). Tbiliss. Gos. Univ. Trudy Ser. Meh.-Mat. Nauk 84 (1962), 77-92.

D. $(W^{(\alpha)}, \lambda, \gamma)$ and $(H^{(\alpha)}, \lambda, \gamma)$ methods

Slepenčuk, K.M., Tauberian theorems for certain methods of summation of double series.(Russian) Izv. Vysš. Včebn. Zaved. Matematika 1964, no. 6(43), 153-158.

E. <u>$(H^{(\alpha)}, \lambda)$ methods</u>

Slepenčuk, K.M., A theorem of Tauberian type for $(H^{(\alpha)}, \lambda)$ methods of summation of double series. (Ukrainian, Russian and English summaries), Dopovidi Akad. Nauk Ukrain. RSR, (1964), 312-314.

2. Convergence Factors

As is pointed out by C. N. Moore[3]; "all methods for summing a divergent series which have come into general use may be classified as mean-value methods or convergence factors methods. Corresponding to any method of either type there can be constructed a formally equivalent method of the other type. The range of validity of the corresponding methods is in general approximately the same ..."

In view of the above remark and the fact that the various mean-value methods of summation have been considered rather extensively in Chapter III, we shall not discuss the results of this method in detail.

Convergence factors may be classified into two types. The set of convergence factors that preserve convergence for a convergent series or produce convergence for a summable series when introduced into the terms of such a series is said to be of type I, and the set of convergence factors used in defining the sum of a series is said to be of type II. Convergence factors of type II have all the properties

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(3) Moore, C.N., Summable series and convergence factors, A.M.S. Colloquium Publ. 22(1938) p. iii.

of type I and some additional ones. For a detail study of the application of the two types of convergence factors to convergent and summable series, the reader is referred to [15, 1938].

We shall state, as an example, a theorem for convergence factors of type I.

Let

$$\sum_{i,j=0}^{\infty} u_{ij} \tag{4.2.1}$$

be a double series and let

$$f_{ij}(\alpha) \quad , \quad i, j = 0, 1, 2, \dots \quad , \tag{4.2.2}$$

be a doubly infinite set of functions defined on a set of points, $E(\alpha)$, in a space of any number of dimensions, with co-ordinates real or complex. Form the generalized series

$$\sum_{i,j=0}^{\infty} u_{ij} f_{ij}(\alpha) \quad . \tag{4.2.3}$$

<u>Definition 4.2.1</u>

The functions $f_{ij}(\alpha)$ are said to be convergence factors of type I if the series $\sum u_{ij} f_{ij}(\alpha)$ is convergent whenever the series $\sum u_{ij}$ is convergent.

Suppose the partial sums

$$s_{mn} = \sum_{i,j=0}^{m,n} u_{ij} \tag{4.2.4}$$

of the double series (4.2.1) is bounded, i.e.

$$|s_{mn}| < C \quad \text{for some constant } C . \tag{4.2.5}$$

Let

$$\sigma_{pq}(\alpha) = \sum_{i,j=0}^{p,q} u_{ij}\, f_{ij}(\alpha) \tag{4.2.6}$$

be the generalized partial sums and let

$$\Delta_{10}\, f_{ij} = f_{ij} - f_{i+1,j} \ ,$$

$$\Delta_{01}\, f_{ij} = f_{ij} - f_{i,j+1} \ ,$$

$$\Delta_{11}\, f_{ij} = \Delta_{10}(\Delta_{01}\, f_{ij}) = \Delta_{01}(\Delta_{10}\, f_{ij})$$

$$= f_{ij} - f_{i+1,j} - f_{i,j+1} + f_{i+1,j+1} \ ,$$

whence we have

<u>Theorem 4.2.1</u>

A necessary and sufficient condition that the series (4.2.3) converge in $E(\alpha)$, and that for each α, its partial sums $\sigma_{pq}(\alpha)$ remain bounded for all (p, q), whenever the series (4.2.1) is convergent and satisfies (4.2.5) are that the functions (4.2.2) satisfy

(i) $\sum_{i,j=0}^{\infty} |\Delta_{11} f_{ij}(\alpha)| < K(\alpha)$, $K(\alpha)$ a positive function of $\alpha \in E(\alpha)$

(ii) $\lim_{n \to \infty} \Delta_{10} f_{in}(\alpha) = 0$, $\alpha \in E(\alpha)$; for all i ,

(iii) $\lim_{m \to \infty} \Delta_{01} f_{mj}(\alpha) = 0$, $\alpha \in E(\alpha)$; for all j .

Theorems for the application of convergence factors of type I and type II to restrictedly and regularly convergent series as well as Nörlund summable series have been formulated by Moore. For a detailed study of these theorems and other results related to convergence factors, the reader is referred to [7, 1933], [10, 1912], [11, 1913], [12, 1927], [13, 1934], [14, 1936] and [15, 1938].

3. Summability Factors

The term summability factors has an analogous meaning to that of convergence factors as do its applications. Up to the present date, there has not been much work done on this topic. The known literature written on it is dated 1957 by G. Kangro. Other authors are S. N. Maheshwari and S. Baron. We list, in chronologic order, all the references available.

i. Kangro, G., On summability factors for double series. Uč. Zap. Tartu. Gos. Univ. 46(1957), 3-42 (Russian. Estonian summary).

ii. Kangro, G., and Baron, S., Summability factors for double series summable by Cesàro's method. Dokl. Akad. Nauk, SSSR 124(1959), 751-753 (Russian).

iii. Kangro, G., and Baron, S., Summability factors for Cesàro-summable and Cesàro-bounded double series. Tartu Rükl. Ül. Toimetised 73(1959), 3-49 (Russian).

iv. Baron, S., Summability factors for double series which are summable or bounded by Cesàro methods of real order. (Russian. Estonian and German summaries), Tartu Rükl. Ül. Toimetised 102(1961), 91-117.

v. Baron, S., Summability factors and summability for double series which are absolutely Cesàro-summable. (Russian. Estonian and German summaries) Tartu Rükl. Ül. Toimetised 102(1961), 118-134.

vi. Baron, S., Absolute summability factors for Cesàro-summable and Cesàro-bounded double series. (Russian. Estonian and German summaries) Tartu Rükl. Ül. Toimetised 102(1961), 135-155.

vii. Maheshwari, S. N., On the absolute summability factors of double infinite series. Boll. Un. Mat. Ital. (3) 16(1961), 367-378.

viii. Kangro, G. and Baron, S., Factors of summability and absolute summability for double series which are absolutely summable by weighted means of Riesz. (Russian. Estonian and German summaries). Tartu Rükl. Ül. Toimetised 129 (1962), 155-169.

ix. Baron, S., Summability factors for double series which are summable or bounded by the weighted means method of Riesz. (Russian. Estonian and English summaries). Tartu Rükl. Ül. Toimetised 129(1962), 225-240.

4. Multiplication of Double Series

The convergence of the Dirichlet product and Cauchy product of double series was treated in Chapter I. In Chapter III, we stated the conditions for the Rieszian summability of the Dirichlet product of Rieszian summable series. We give presently the conditions for the restricted Cesàro (C, 1 ,1)-summability of the Cauchy product series of double series and for the Abel power series method of summability for such series. Following these results, a list of the literature written on multiplication of double series is given for reference.

Let the series $\sum' c_{mn}$ be the Cauchy product of two given series $\sum' a_{ij}$ and $\sum b_{pq}$ where

$$c_{mn} = \sum_{i+p=m} \sum_{j+q=n}' a_{ij} \; b_{pq} \quad .$$

Then we have

<u>Theorem 4.4.1</u> [3, 1943]

If $\sum a_{ij}$ and $\sum b_{pq}$ converge to A and B respectively, and

$$\lim_{i+j \to \infty} a_{ij} = 0 \quad , \quad \lim_{p+q \to \infty} b_{pq} = 0 \quad ,$$

then $\sum c_{mn}$ is restrictedly summable $(C, 1,1)$ to $C = AB$.

<u>Theorem 4.4.2</u>

If $\sum a_{ij}$ and $\sum b_{pq}$ converge restrictedly to A and B, respectively, and if there exist constants P , Q , R , $\sigma(<1)$ and $\tau(<1)$ such that the partial sums

$$A_{mn} = \sum_{i,j=1}^{m,n} a_{ij} \quad \text{and} \quad B_{mn} = \sum_{p,q=1}^{m,n} b_{pq}$$

satisfy the condition that

$$A_{mn} \, , \, B_{mn} < P + Q[(m+1)/(n+1)]^{\sigma} + R[(n+1)/(m+1)]^{\tau}$$

then $\sum c_{mn}$ is restrictedly summable $(C, 1, 1)$ to $C = AB$.

Remark 4.4.1

Čelidze [2, 1953] obtains the result of Theorem 4.4.2 by requiring that the partial sums A_{mn} be bounded and $B_{mn}/m+n \to 0$ as $m, n \to \infty$.

Theorem 4.4.3

If $\sum a_{ij}$ and $\sum b_{pq}$ converge to A and B respectively, and have partial sums A_{mn} and B_{mn} for which

$$\lim_{m+n \to \infty} A_{mn}/m+n = 0 \;, \quad \lim_{m+n \to \infty} B_{mn}/m+n = 0 \;,$$

then $\sum c_{mn}$ is restrictedly summable to $C = AB$ by the Abel power series method.

i. Cesari, L., On multiplication of double series. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1(1946), 289-292.

ii. Kull', I. G., Multiplication of summable double series. Tartu Rükl. Ül. Toimetised 62(1958), 3-59.

iii. Kull', I. G., Multiplication of double series by Dirichlet's rule. (Russian. Estonian and English summaries) Tartu Rükl. Ül. Toimetised 102(1961), 185-192.

iv. Reiners, È., Mean value theorems and multiplication of double summable series. Tartu Rükl. Ül. Toimetised 73(1959), 50-83 (Russian. Estonian and English summaries).

v. Tret́jakov, V. P., On the theory of formal multiplication of double series. (Russian), Izv. Vysš. Učebn. Zaved. Matematika 1961 no. 5(24), 78-85.

vi. Tret́jakov, V. P., Multiplication of double series in the case when the partial sums of one of the series are unbounded. (Russian), Izv. Vysš. Učebn. Zaved. Mat. 1964, 1(38), 122-124.

5. Results Related to Double Sequences and Series

We list, as our last section, the references of all the articles which are directly concerned with double sequences and series and have not been included in the previous chapters in order to maintain the coherence of each chapter.

A. Saalschutzian theorems for basic double series

Al-Salam, W. A., Saalschutzian theorems for basic double series, J. London Math. Soc. 40(1965), 455-458.

B. Representation of measurable functions

Dzagnidze, O. P., Representation of measurable functions of two variables by double series.(Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin.SSR 34(1964), 277-282.

C. On universal double series

Dzagnidze, O. P., On universal double series. (Russian. Georgian summary) Soobšč. Akad. Nauk Gruzin. SSR 34(1964), 525-528.

D. Ordinary and iterated limits

Garcia, P. J., Relation between the ordinary and iterated limits of doubly infinite sequences. Gaceta Mat. (1) 5 (1953), 8-10. (Spanish).

E. Evaluation of double sums and integrals

Pospeev, V. E., On the evaluation of double sums and integrals. (Russian. Uzbek summary) Izv. Akad. Nauk. Uzssr Ser. Fiz. - Mat. Nauk 1963, no. 4, 39-45.

F. Mean value theorem for double series

Reĭmers, È. G., Mean value theorems for double series, Tartu Rükl. Ül. Toimetised 62(1958), 60-79. (Russian. Estonian and English summaries).

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